Model Predator-Prey Orde Fractional

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Introduction to Fractional Calculus*

I. Special functions

Gamma function:

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$$

$$\Gamma(1)=\Gamma(2)=1$$

$$\Gamma(x+1)=x\Gamma(x)$$

$$\Gamma(n)=(n-1)!;\ n\in N$$

$$\Gamma(1/2)=\sqrt{\pi}$$

$$\Gamma(n+1/2)=\frac{\sqrt{\pi}}{2^n}(2n-1)!;\ n\in N$$

$$\exp(x)=\sum_{j=0}^{\infty}\frac{x^j}{j!}=\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+1)}$$

$\alpha = 1$ ↑ ↓ **Mittag-Leffler function:**

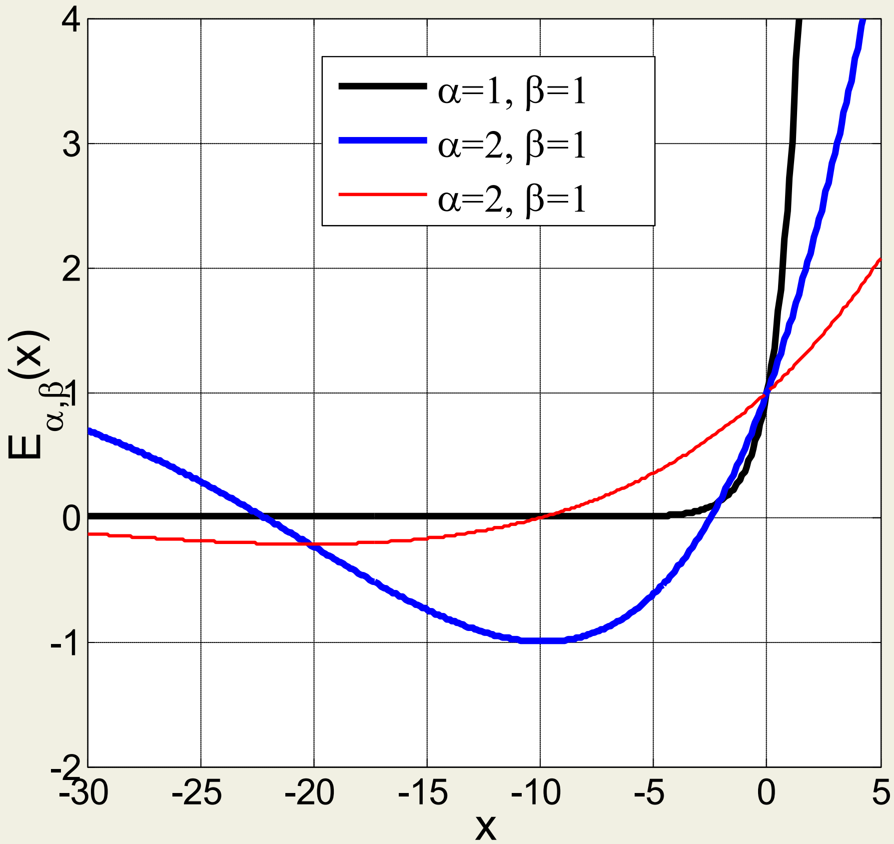
$$E_\alpha(x)=\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\alpha j+1)};\ \alpha>0,\ \alpha\in R,\ x\in C$$

Two-parameter generalization:

$$E_{\alpha,\beta}(x)=\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\alpha j+\beta)};\ \alpha,\beta>0,\ \alpha,\beta\in R,\ x\in C$$

I. Petras, *Fractional-order nonlinear systems: Modeling, Analysis and Simulation*, Springer, 2011.

Two-parameter Mittag-Leffler:



$$E_{\alpha,1}(x) = E_\alpha(x)$$

$$E_{1,1}(x) = \exp(x)$$

$$E_{2,1}(x^2) = \sum_{j=0}^{\infty} \frac{x^{2j}}{\Gamma(2j+1)} = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} = \cosh(x)$$

$$E_{2,2}(x^2) = \sum_{j=0}^{\infty} \frac{x^{2j}}{\Gamma(2j+2)} = \frac{1}{x} \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} = \frac{\sinh(x)}{x}$$

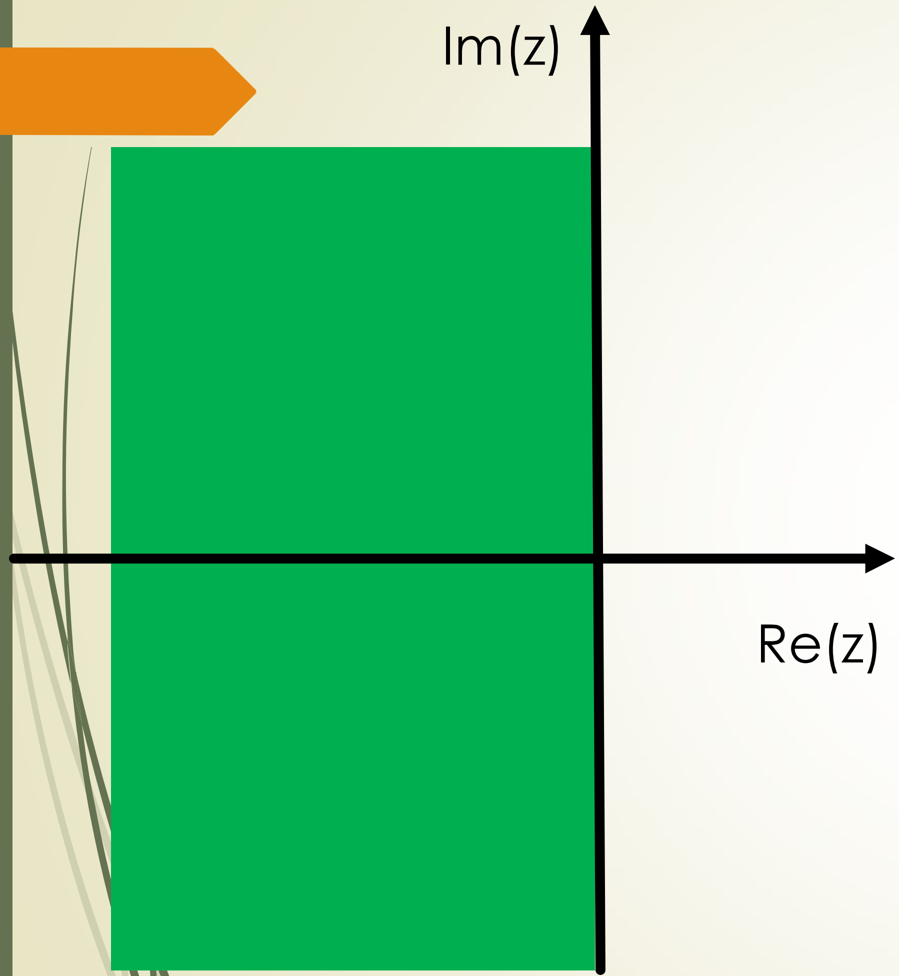
Convergence of Mittag-Leffler Function

Let $\alpha > 0$. The Mittag-Leffler function $E_\alpha(z)$, where $z = re^{i\theta}$, $r = |\lambda|$, $\theta = \arg(z)$, behaves as follows

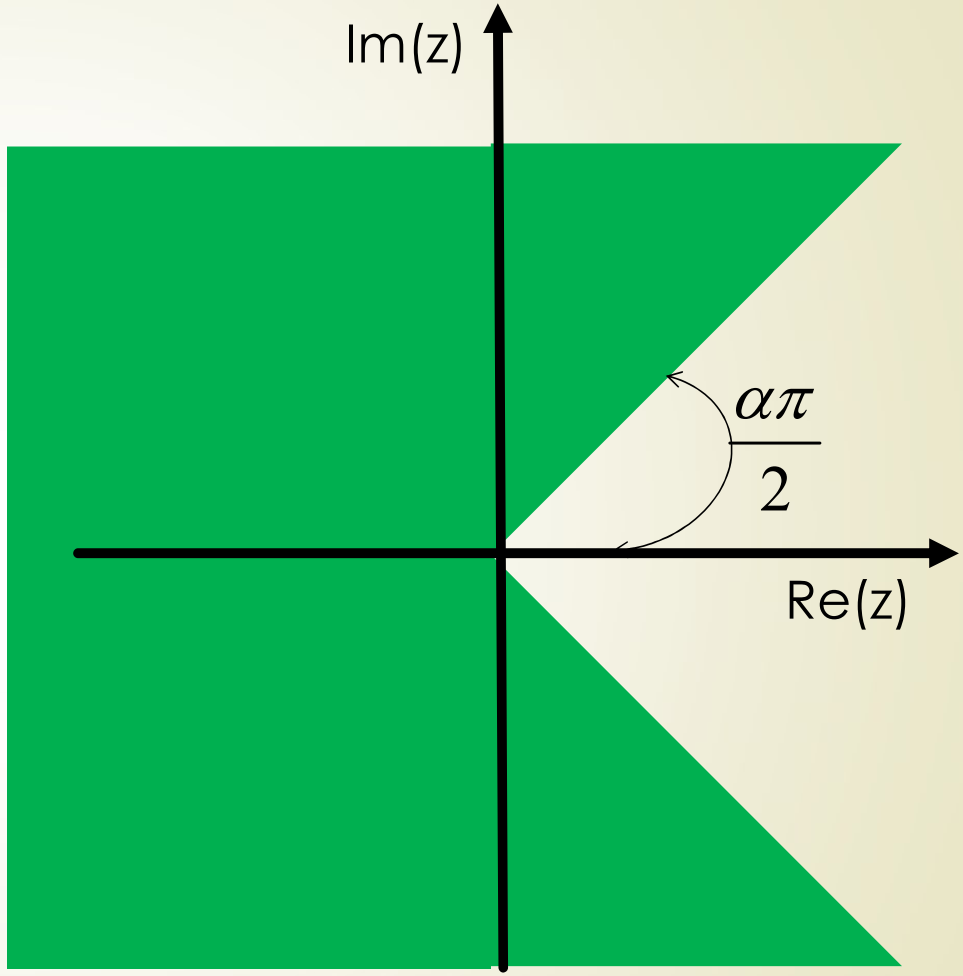
1. $E_\alpha(re^{i\theta}) \to 0$ for $r \to \infty$ if $|\theta| > \frac{\alpha\pi}{2}$,
2. $E_\alpha(re^{i\theta})$ remains bounded for $r \to \infty$ if $|\theta| = \frac{\alpha\pi}{2}$,
3. $E_\alpha(re^{i\theta}) \to \infty$ for $r \to \infty$ if $|\theta| < \frac{\alpha\pi}{2}$,

Remark:

For classical $\alpha = 1$, Mittag-Leffler is reduced to exp(z). As, $|z| \to \infty$ then $\exp(z)$ (a) goes to zero as $|\arg(z)| > \pi/2$, (b) remains bounded if $|\arg(z)| = \pi/2$, (c) grows without bound if $|\arg(z)| < \pi/2$.



$E_1(z) = \exp(z)$



$E_\alpha(z)$

Introduction to Fractional Calculus*

Definition 1. Let $\alpha \in (0, \infty)$, the operator J^{α} defined by

$$J^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \qquad (t \in [0, a])$$

is called **the Riemann-Liouville fractional integral operator of order α**, where $J^0 = Id$ is the identity operator.

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Definition 2. Let $\alpha \in (0, \infty)$ and $m = [\alpha]$, where $[t] \coloneqq \min\{k \in \mathbb{Z} : k \geq \alpha\}$, the operator D^{α} defined for u by

$$D^{\alpha} u(t) \coloneqq \frac{d^m}{dt^m} J^{m-\alpha} u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} u(\tau) d\tau,$$

is called **the Riemann-Liouville fractional differential operator of order α**.

I. Petras, Fractional-order nonlinear systems: Modeling, Analysis and Simulation, Springer, 2011.

Introduction to Fractional Calculus

Definition 3. Assume that $\alpha \geq 0$ and u is such that $J^{m-\alpha}u^{(m)}$ exists, where $m = \lceil \alpha \rceil$. **the Caputo fractional differential operator of order α** is defined by

$$D_*^\alpha u(t) := J^{m-\alpha}u^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}u^{(m)}(\tau)d\tau$$

The Reimann-Liouville vs. Caputo fractional differential operator:

$$D_*^\alpha u(t) = D^\alpha u(t) - \sum_{v=0}^{m-1} r_v^\alpha(t)u^{(v)}(0)\,; \quad r_v^\alpha(t) = \frac{t^{v-\alpha}}{\Gamma(v+1-\alpha)}$$

For the case $m = 1 \text{ or } 0 < \alpha < 1$, then "correction term reads"

$$D_*^\alpha u(t) = D^\alpha u(t) - r_0^\alpha(t)y_0$$

Example:

$$a = 0,\ \alpha = 1/2\ (m = 1), u(t) = t$$

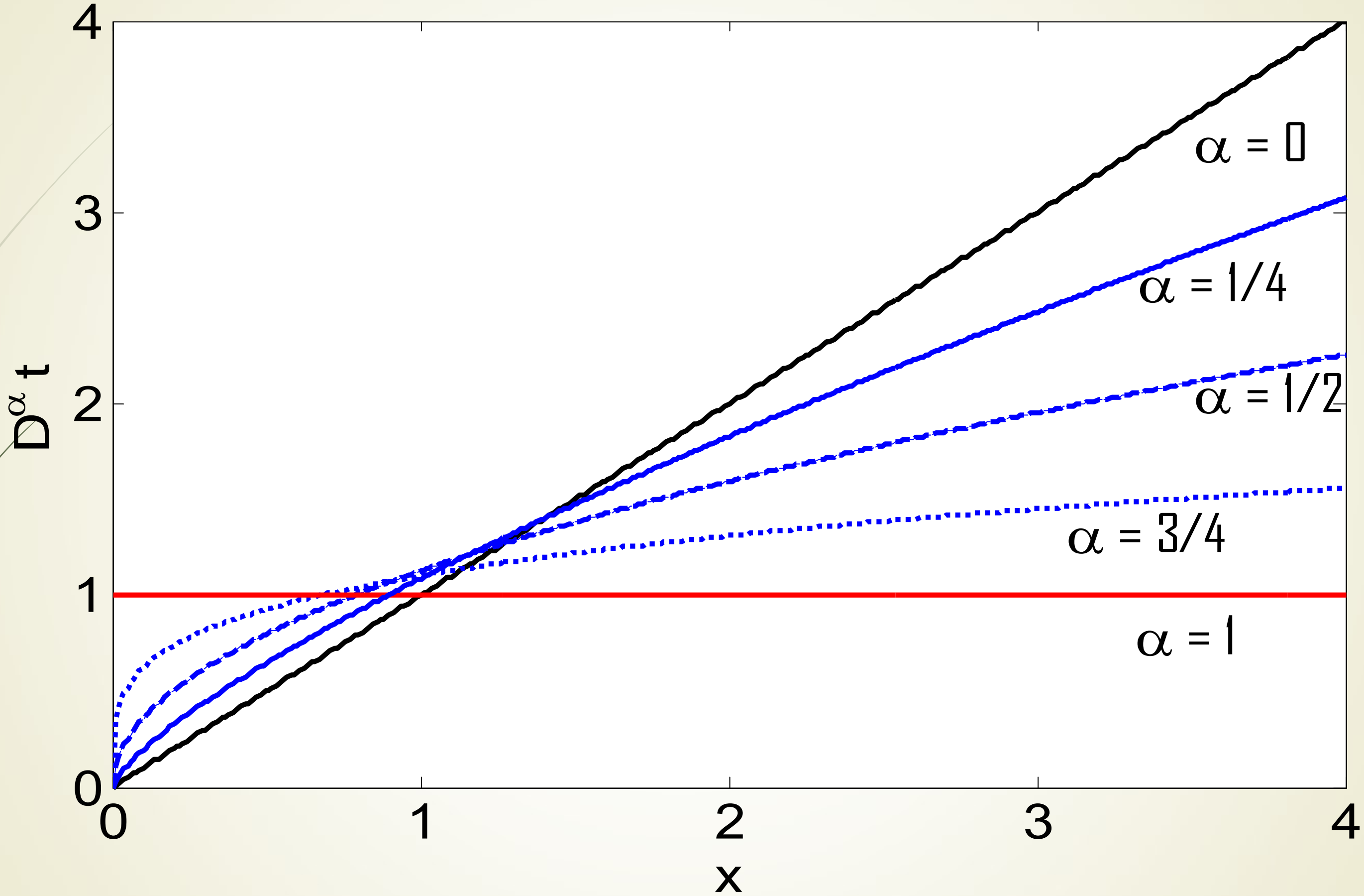
$$D_*^{1/2} t = \frac{1}{\Gamma(1/2)} \int_0^t \frac{1}{(t-\tau)^{1/2}} d\tau$$

$$= -\frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} d(t-\tau)$$

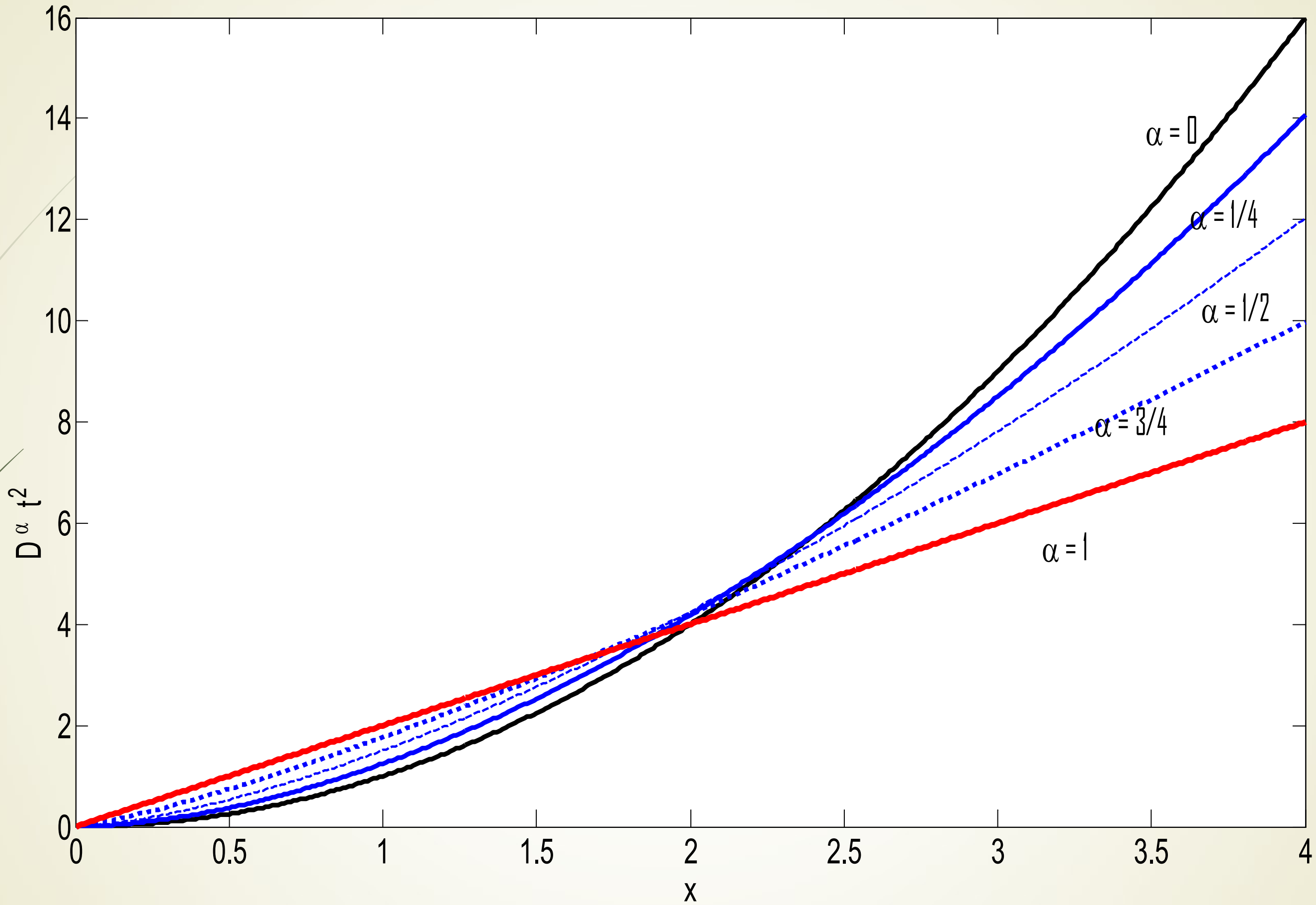
$$= -\frac{1}{\sqrt{\pi}} \int_{\sqrt{t}}^{0} \frac{1}{u} du^2$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} 2du = \frac{2\sqrt{t}}{\sqrt{\pi}}$$

	$D_*^{\alpha} f(t)$	$D_*^{1/3} f(t)$	$D_*^{1/2} f(t)$
$f(t) = const$	0	0	0
$f(t) = t$	$\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}$	$1.1077\, t^{2/3}$	$1.1284\, t^{1/2}$
$f(t) = t^2$	$\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}$	$1.3293\, t^{5/3}$	$1.5045\, t^{3/2}$
$f(t) = t^3$	$\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha}$	$1.4954\, t^{8/3}$	$1.8054\, t^{5/2}$
$f(t) = t^4$	$\frac{24}{\Gamma(5-\alpha)} t^{4-\alpha}$	$1.6314\, t^{11/3}$	$2.0633\, t^{7/2}$
$f(t) = t^5$	$\frac{120}{\Gamma(6-\alpha)} t^{5-\alpha}$	$1.7479\, t^{14/3}$	$2.2926\, t^{9/2}$
$f(t) = t^{1/2}$	$\frac{\sqrt{\pi}}{2\,\Gamma(3/2-\alpha)} t^{1/2-\alpha}$	$0.9553\, t^{1/6}$	0.8862
$f(t) = t^{3/2}$	$\frac{3\sqrt{\pi}}{4\,\Gamma(5/2-\alpha)} t^{3/2-\alpha}$	$1.2282\, t^{7/6}$	$1.3292\, t$
$f(t) = e^t$	$t^{n-\alpha} E_{1,n-\alpha+1}(t)$	$t^{2/3} E_{1,5/3}(t)$	$t^{1/2} E_{1,3/2}(t)$



$$D_*^{\alpha} t = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}, \; 0 < \alpha < 1$$



$$D_*^{\alpha} t^2 = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, \; 0 < \alpha < 1$$

Fractional Initial Value Problem

The solution of linear initial value problem (IVP):

$$D_*^\alpha u(t) = \lambda u(t); \quad t > 0,\ m-1 < \alpha < m,$$
$$u^{(k)}(0) = b_k,\ b_k \in R,\ k = 0,1,\ldots,m-1$$

is given by

$$u(t) = \sum_{k=0}^{m-1} b_k t^k E_{\alpha,k+1}\left(\lambda t^\alpha\right)$$

where $E_{\alpha,\beta}(x)$ is the two-parameter function of Mittag-Leffler type.

Proof: use Laplace transform, see Podlubny, Subsection 1.4

I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999

Fractional Initial Value Problem

$$D_*^{\alpha} u(t) = \lambda u(t); \quad t > 0,\ 0 < \alpha < 1,$$
$$u(0) = b_0,\ b_0 \in R$$

$$u(t) = b_0 E_{\alpha,1}(\lambda t^{\alpha}) = b_0 E_{\alpha}(\lambda t^{\alpha})$$
$$= b_0 \sum_{k=0}^{\infty} \frac{(\lambda t^{\alpha})^j}{\Gamma(\alpha k + 1)}$$

- Suppose $\alpha > 0$ and $\lambda = re^{i\theta}$ where $r = |\lambda|, \theta = arg(\lambda)$.

1. $E_{\alpha}(re^{i\theta}) \to 0$ as $r \to \infty$ if $arg(\lambda) = |\theta| > \frac{\alpha\pi}{2}$,
2. $E_{\alpha}(re^{i\theta})$ remains bounded as $r \to \infty$
   if $arg(\lambda) = |\theta| = \frac{\alpha\pi}{2}$,
3. $E_{\alpha}(re^{i\theta}) \to \infty$ as $r \to \infty$ if $arg(\lambda) = |\theta| < \frac{\alpha\pi}{2}$

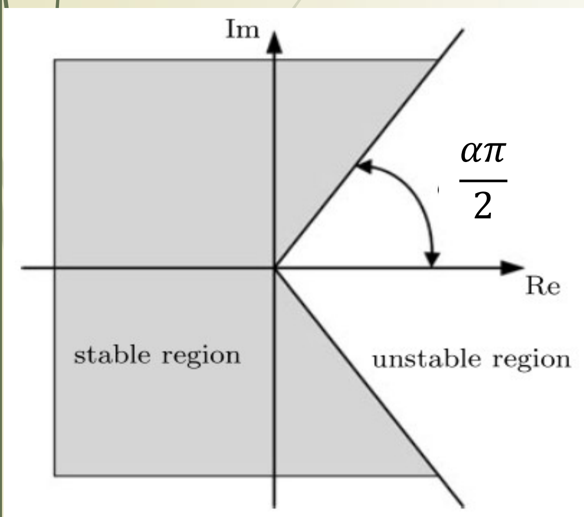
Fractional (system) Initial Value Problem

$$D_*^{\alpha}\vec{u}(t) = A\vec{u}(t), \mathrm{t} > 0, 0 < \alpha, \vec{u} \in \mathbb{R}^n, A_{n\times n}$$
$$\vec{u}(0) = \vec{b}_0, \vec{b}_0 \in \mathbb{R}^n$$

$$B^{-1}AB = C$$
$$AB = BC$$
$$A = BCB^{-1}$$

$$D_*^{\alpha}\vec{u}(t) = A\vec{u}(t) = (BCB^{-1})\vec{u}(t)$$
$$D_*^{\alpha}(B^{-1}\vec{u}(t)) = C\left(B^{-1}\vec{u}(t)\right)$$
$$D_*^{\alpha}(\vec{v}(t)) = C\left(\vec{v}(t)\right); \vec{v}(t) = B^{-1}\vec{u}(t)$$

Autonomous Nonlinear Fractional System



Theorem 1*. Consider the following autonomous nonlinear fractional-order system

$$D_*^{\alpha}\vec{u} = \vec{f}(\vec{u}); \quad \vec{u}(0) = \vec{u}_0; \;\; 0 < \alpha < 1.$$

The equilibrium points of the above system are solutions to the equation $\vec{f}(\vec{u}) = 0$. An equilibrium is locally asymptotically if all eigenvalues λ_j of the Jacobian matrix $J = \frac{\partial \vec{f}}{\partial \vec{u}}$ at the equilibrium satisfy $\left|\arg(\lambda_j)\right| > \frac{\alpha\pi}{2}$.

*D. Matignon, Stability results for fractional differential equations with application to control processing, in: *Appl. Computational Eng. Sys.* 2, France, 1996, pp. 963-968

I. Petras, *Fractional-order nonlinear systems: Modeling, Analysis and Simulation*, Springer, 2011.

Ecology

Prey → Predator

Prey → Refuge

Additional Food → Predator

Mathematical Model (1)

Ghosh et al (2017)*:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')xy}{1+\theta\xi+x}$$

$$\frac{dy}{dt} = \frac{\beta[(1-c')x+\xi]y}{1+\theta\xi+x} - \delta y$$

The growth rate depends instantly on the current state !

How to include the previous history (memory effects)?

⬇

Fractional-order derivative

*Ghosh, J., B. Sahoo, and S. Poria. 2017, Prey-predator dynamics with prey refuge providing additional food to predator, *Chaos, Solitons and Fractals,* 96: 110–119.

Mathematical Model (2)

$$D_*^{\alpha} x = x\left(1 - \frac{x}{\gamma}\right) - \frac{(1 - c')xy}{1 + \theta\xi + x}$$

$$D_*^{\alpha} y = \frac{\beta[(1 - c')x + \xi]y}{1 + \theta\xi + x} - \delta y$$

Non-negativity of solution

Lemma 1. [1]

Let $0 < \alpha \leq 1, u(t) \in C[a,b]$ and $D_*^\alpha u(t) \in C[a,b]$. Then the following statements hold:

1. If $D_*^\alpha u(t) \geq 0, \forall t \in (a,b)$, then $u(t)$ is a non-decreasing function for all $t \in [a,b]$
2. If $D_*^\alpha u(t) \leq 0, \forall t \in (a,b)$, then $u(t)$ is a non-increasing function for all $t \in [a,b]$

Theorem 2.

All solution of model (2) with initial condition $x(0) \geq 0, y(0) \geq 0$ are non-negative.

Proof:

We will prove that $x(t) \geq 0, y(t) \geq 0$ for all $t \geq 0$. Suppose this is not true, then there is a constant $t_1 > 0$ such that $x(t) > 0$ for $0 \leq t < t_1$; $x(t_1) = 0$; and $x(t_1^+) < 0$. Substituting $x(t_1) = 0$ into model (2) gives

$$D_*^\alpha x(t_1)\Big|_{x(t_1)=0} = 0.$$

Since $D_*^\alpha x(t_1) = 0$, Lemma 1 says that $x(t_1^+) = 0$ which contradicts with $x(t_1^+) < 0$. Hence $x(t) \geq 0$ for all $t \geq 0$. Similar argument can be used to prove that $y(t) \geq 0$ for all $t \geq 0$.

Boundedness of solution

Lemma 3.[2]
Let $u(t)$ be a continuous function on $[t_0, +\infty)$ and satisfying

$$D_*^\alpha u(t) \leq -\lambda u(t) + \mu;\ u(t_0) = u_0$$

where $0 < \alpha < 1, (\lambda, \mu) \in \mathbb{R}^2$ and $\lambda \neq 0$, and $t_0 \geq 0$ is the initial time.
Then $u(t) \leq \left(u_0 - \frac{\mu}{\lambda}\right) E_\alpha[-\lambda(t - t_0)^\alpha] + \frac{\mu}{\lambda}$.

Theorem 4.
Solution of model (2) with initial condition $x(0) \geq 0, y(0) \geq 0$ is uniformly bounded.

Proof:
First define $w(t) = x(t) + \frac{1}{\beta} y(t)$ and $\sigma(x) = 1 + \theta\xi + x$ such that

$$D_*^\alpha w(t) + (\beta\xi - \delta) w(t) = x - \frac{1}{\gamma} x^2 + (\beta\xi - \delta) x + \left(\frac{1 - \sigma(x)}{\sigma(x)}\right) \xi y$$

$$\leq -\frac{1}{\gamma}\left(x - \frac{\gamma(1+\beta\xi-\delta)}{2}\right)^2 + \frac{\gamma(1+\beta\xi-\delta)^2}{4}$$

$$\leq \frac{\gamma(1+\beta\xi-\ \ )^2}{4}$$

Boundedness of solution

Using Lemma 3, we have

$$w(t) \leq \left(w(0) - \frac{\gamma(1+\beta\xi-\delta)^2}{4(\beta\xi-\delta)} \right) E_\alpha[-(\beta\xi-\delta)t^\alpha] + \frac{\gamma(1+\beta\xi-\delta)^2}{4(\beta\xi-\delta)}$$

For $t \to \infty$, we have $w(t) \to \frac{\gamma(1+\beta\xi- \quad)^2}{4(\beta\xi-\delta)}$.

Hence, all solutions which start from $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+\}$ are confined to the region

$$\Gamma = \left\{ (x,y) \in \Omega | x + \frac{1}{\beta} y \leq \frac{\gamma(1+\beta\xi- \quad)^2}{4(\beta\xi-\delta)} + \epsilon, \epsilon > 0 \right\}.$$

Lemma 5. [3]

Existence and Uniqueness of solution

Consider a fractional-order differential system

$$D_*^{\alpha} u(t) = f\big(t, u(t)\big), t > 0$$

with initial condition $x(0) \geq 0, y(0) \geq 0$ and $0 < \alpha < 1$, $f: (0, \infty) \times \Psi \to \mathbb{R}^2, \Psi \subseteq \mathbb{R}^2$. If $u(t)$ satisfies the Lipschitz condition w.r.t u, then there exists a unique solution of the above system on : $(0, \infty) \times \Psi$.

Theorema 6.

Consider model (2) with initial condition $x(0) \geq 0, y(0) \geq 0$ and $0 < \alpha < 1$, $f: (0, \infty) \times \Omega_M \to \mathbb{R}^2$, where $\Omega_M = \{(x, y) \in \mathbb{R}_+^2 | max\{|x|, |y|\} \leq M\}$ for sufficiently large M. This IVP has a unique solution.

Proof:

Consider $H(X) = \big(H_1(X), H_2(X)\big)$ with

$$H_1(X) = x\left(1 - \frac{x}{\gamma}\right) - \frac{(1 - c')xy}{1 + \theta\xi + x};\ H_2(X) = \frac{\beta[(1 - c')x + \xi]y}{1 + \theta\xi + x} - \delta y$$

For any , $X = (x, y), \bar{X} = (\bar{x}, \bar{y}), X, \bar{X} \in \Omega_M$, we can show that

$$\|H(X) - H(\bar{X})\| \leq L\|X - \bar{X}\|, L \equiv L(\gamma, c', \theta, \beta, M)$$

Equilibrium Point and Stability

$$D_*^{\alpha}x(t) = D_*^{\alpha}y(t) = 0$$

$$x\left[\left(1-\frac{x}{\gamma}\right)-\frac{(1-c')y}{1+\theta\xi+x}\right]=0$$

$$y\left[\frac{\beta[(1-c')x+\xi]}{1+\theta\xi+x}-\delta\right]=0.$$

1. Extinction point: $E_0 = (0,0)$
2. Predator extinction point: $E_1 = (\gamma, 0)$
3. Coexistence point: $E_* = (x^*, y^*)$ where $x^* = \frac{\delta+(\delta\theta-\beta)\xi}{\beta(1-c')-\delta}$ and $y^* = \left(1-\frac{x^*}{\gamma}\right)\left(\frac{1+\theta\xi+\ ^*}{(1-c')}\right)$. E_* exists if $x^* < \gamma$ and $\beta(1-c') < \delta < \frac{\beta\xi}{1+\theta\xi}$.

Local Stability

Jacobian matrix at equilibrium (x^*, y^*)

$$J(x^*, y^*) = \begin{bmatrix} 1 - \frac{2x^*}{\gamma} - \frac{(1-c')(1+\theta\xi)y^*}{(1+\theta\xi+x^*)^2} & \frac{-(1-c')x^*}{1+\theta\xi+x^*} \\ \frac{\beta y^*\left[(1-c')(1+\theta\xi)-\xi\right]}{(1+\theta\xi+x^*)^2} & \frac{\beta\left[(1-c')x^*+\xi\right]}{1+\theta\xi+x^*} - \delta \end{bmatrix}$$

Theorem 8.

1. Equilibrium point E_0 is unstable
2. Equilibrium point E_1 is asymptotically stable if $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+} < \delta$

At $E^* = (x^*, y^*)$, the Jacobian matrix has a characteristics equation

$$\lambda^2 - a_1\lambda + a_2 = 0,$$

$$a_1 = \left[\frac{x^*}{\gamma}\left(\frac{\gamma - x^*}{1+\theta\xi+\ {}^*} - 1\right)\right],\ a_2 = \frac{\beta(1-c')[(1-c')(1+\theta\xi)-\xi]\ {}^*y^*}{(1+\theta\xi+\ {}^*)^3}$$

$$\lambda_{1,2} = \frac{1}{2}\left(a_1 \pm \sqrt{D}\right),\ D = (a_1)^2 - 4a_2.$$

Theorem 9.

Equilibrium E^* is asymptotically stable if one of the following mutually exclusive conditions holds:

(i) $a_1 < 0$; $a_2 > 0$ and $D \geq 0$

(ii) $D < 0$ and $\left|\sqrt{D}/a_1\right| > \tan(\frac{\alpha\pi}{2})$

Proof.

(i) If $a_1 < 0$; $a_2 > 0$ and $D \geq 0$ then $\lambda_{1,2} < 0$, hence $\arg(\lambda_{1,2}) = \pi > \frac{\alpha\pi}{2}$ and the result follows.

(ii) Suppose $D < 0$. If λ is an eigenvalue, then $\bar{\lambda}$ is also an eigenvalue. Using $\left|\sqrt{D}/a_1\right| > \tan(\frac{\alpha\pi}{2})$, we have that $\left|\frac{\lambda-\bar{\lambda}}{\lambda+\bar{\lambda}}\right| = \left|\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right| = |\arg(\lambda)| = \left|\sqrt{D}/a_1\right| > \tan(\frac{\alpha\pi}{2})$. Therefore the stability of E^* follows. □

Global Stability

Lemma 10. [4]
Let $x(t) \in \mathbb{R}_+$ be continuous and derivable function. Then, for any $t > t_0$ and $\alpha \in (0,1)$:

$$D_*^\alpha \left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*} \right] \leq \left(1 - \frac{x^*}{x(t)} \right) D_*^\alpha x(t), x^* \in \mathbb{R}_+$$

Lemma 11. ***Generalized Lasalle Invariance Principle*** **[5]**
Suppose D is a bounded closed set and every solution of

$$D_*^\alpha x(t) = f(x)$$

starts from a point and remains in D for all time. *If* $\exists V(x): D \to \mathbb{R}$ with continuous first partial derivatives satisfies

$$D_*^\alpha V|_{D_*^\alpha x(t) = f(x)} \leq 0.$$

Let $E = \left\{ x | D_*^\alpha V|_{D_*^\alpha x(t) = f(x)} = 0 \right\}$ and M be the largest invariant set of E. Then every solution $x(t)$ originating in D tends to M as $t \to \infty$.

Theorem 12.

E_1 is globally asymptotically stable if $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+} \leq \delta$.

Proof:

1. Define a Lyapunov function $U(x,y) = \left(x - \gamma - \gamma \ln\frac{x}{\gamma} + \frac{y}{\beta}\right)$, then

$$D_*^\alpha U(x,t) \leq \frac{x-\gamma}{x} D_*^\alpha x(t) + \frac{1}{\beta} D_*^\alpha y(t) = -\frac{1}{\gamma}(x-\gamma)^2 + \left[\frac{(1-c')\gamma+\xi}{1+\theta\xi+x} - \frac{\delta}{\beta}\right] y$$

If $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+x} \leq \delta$ then $D_*^\alpha U(x,t) \leq 0$ for all $(x,t) \in \mathbb{R}_+^2$. Furthermore $D_*^\alpha U(x,t) = 0$ implies that $x = \gamma$ and $y = 0$. Hence, the only invariant set on which $D_*^\alpha U(x,t) = 0$ is singleton $\{E_1\}$.

Lasalle Invariance Principle → E_1 is globally stable

Theorem 13.

E^* is globally asymptotically stable in the region $\Omega = \left\{(x, y): \frac{y}{y^*} > \frac{x}{x^*} > 1\right\}$.

Proof:

1. Define a Lyapunov function

$$V(x, y) = \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + \frac{1}{\beta}\left(y - y^* - y^* \ln \frac{y}{y^*}\right)$$

then

$$D_*^\alpha V(x, t) \leq \frac{x - x^*}{x} D_*^\alpha x(t) + \frac{1}{\beta}\left(\frac{y - y^*}{y}\right) D_*^\alpha y(t)$$

$$= -\frac{1}{\gamma}(x - x^*)^2 - \frac{\xi(x - x^*)(y - y^*)}{(1 + \theta\xi + x)(1 + \theta\xi + x^*)} - \frac{(1 - c')(x - x^*)(x^* y - x y^*)}{(1 + \theta\xi + x)(1 + \theta\xi + x^*)}$$

For any $(x, y) \in \Omega$ we have $D_*^\alpha V(x, t) \leq 0$. Furthermore $D_*^\alpha V(x, t) = 0$ implies that $x = x^*$ and $y = y^*$.

Hence, singleton $\{E^*\}$ is the only invariant set on which $D_*^\alpha U(x, t) = 0$.

→ E^* is globally asymptotically stable

Integer-Order Derivative

$$\frac{du}{dt} = u'(t) = \lim_{h\to 0}\left(\frac{u(t) - u(t-h)}{h}\right)$$

$$\frac{d^2u}{dt^2} = u''(t) \quad = \lim_{h\to 0}\left(\frac{u'^{(t)} - u'^{(t-h)}}{h}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{u(t) - u(t-h)}{h} - \frac{u(t-h) - u(t-2h)}{h}\right)$$

$$= \lim_{h\to 0}\frac{u(t) - 2u(t-h) + u(t-2h)}{h^2}$$

$$\frac{d^3u}{dt^3} = u'''(t)$$

$$= \lim_{h\to 0}\frac{u(t) - 3u(t-h) + 3u(t-2h) - u(t-3h)}{h^3}$$

$$\frac{d^nu}{dt^n} = u^n(t) = \lim_{h\to 0}\frac{1}{h^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}u(t-jh),$$

$$\binom{n}{j} = \frac{n(n-1)(n-2)\ldots(n-j+1)}{j!} = \frac{n!}{j!\,(n-j)!}$$

$$= \frac{\Gamma(n+1)}{\Gamma(j+1)\Gamma(n-j+1)}$$

Grünwald-Letnikov Approximation*

Using the notation of finite difference of an equidistant grid in $[0, t_{n+1}]$, $t_{n+1} \in R$, Grünwald-Letnikov defines

$$D^\alpha u(t_{n+1}) = \lim_{h\to 0} \frac{1}{h^\alpha} \Delta_h^\alpha u(t_{n+1}).$$

$$\frac{1}{h^\alpha} \Delta_h^\alpha u(t_{n+1}) = \frac{1}{h^\alpha} \left(u(t_{n+1}) - \sum_{j=1}^{n+1} c_j^\alpha \, u\big(t_{n+1-j}\big) \right)$$

$$c_j^\alpha = (-1)^{j-1} \binom{\alpha}{j} = (-1)^{j-1} \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}$$

$$(1-z)^\alpha = 1 - \textstyle\sum_{j=1}^{\infty} c_j^\alpha \, z^j;$$

$$c_j^\alpha = \left(1 - \frac{\alpha+1}{j}\right) c_{j-1}^\alpha, \quad c_1^\alpha = \alpha$$

$$0 < c_{n+1}^\alpha < c_n^\alpha < \cdots < c_1^\alpha = \alpha$$

*R. Scherer, S.L. Kalla, Y. Tang, J. Huang, The Grünwald-Letnikov method for fractional differential equations, *Comput. Math. Appl*. **62**, 2011, pp. 902-917

Grünwald-Letnikov Approximation*

Hence

$$D_*^\alpha u(t) \approx \frac{1}{h^\alpha}\Delta_h^\alpha u(t) - r_0^\alpha(t)y_0; \;\; r_0^\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}; 0 < \alpha < 1$$

*R. Scherer, S.L. Kalla, Y. Tang, J. Huang, The Grünwald-Letnikov method for fractional differential equations, *Comput. Math. Appl*. **62**, 2011, pp. 902-917

Persamaan Eksponensial: Skema Eksak

$$\frac{du(t)}{dt} = ru(t);\ u(0) = u_0;\ r \neq 0 \qquad u(t) = u_0 \exp(rt)$$

$$u(t_{n+1}) - u(t_n) = u_0 \exp(r(t_n + h)) - u_0 \exp(r(t_n))$$
$$= u(t_n)(\exp(rh) - 1)$$

$$\frac{u_{n+1} - u_n}{\frac{(\exp(rh) - 1)}{r}} = ru_n$$

V.S. Skema Euler

$$\frac{u_{n+1} - u_n}{h} = ru_n$$

Fungsi Penyebut/ Denominator

Persamaan Logistik: Skema Eksak

$$\frac{du(t)}{dt} = ru(t)(1-u(t));\; u(0)=u_0;\; r>0 \qquad u(t)=\frac{u_0\exp(rt)}{1+u_0(\exp(rt)-1)}$$

$$u(t_{n+1}) = \frac{u_0\exp(rt_{n+1})}{1+u_0(\exp(rt_{n+1})-1)} = \frac{u_0\exp(rt_n)\exp(rh)}{1+u_0(\exp(rt)\exp(rh)-1)}$$

$$= \frac{u_0\exp(rt_n)\exp(rh)}{1+u_0(\exp(rt_n)-1)+u_0\exp(rt_n)(\exp(rh)-1)}$$

$$= \frac{\exp(rh)\left(\dfrac{u_0\exp(rt_n)}{1+u_0(\exp(rt_n)-1)}\right)}{1+\left(\dfrac{u_0\exp(rt_n)}{1+u_0(\exp(rt_n)-1)}\right)(\exp(rh)-1)} = \frac{u(t_n)\exp(rh)}{1+u(t_n)(\exp(rh)-1)}$$

Persamaan Logistik: Skema Eksak

$$u(t_{n+1}) = \frac{u(t_n)\exp(rh)}{1 + u(t_n)(\exp(rh) - 1)}$$

$$u(t_{n+1}) = u(t_n)\exp(rh) - u(t_n)u(t_{n+1})\exp(rh) + u(t_n)u(t_{n+1})$$

$$u(t_{n+1}) - u(t_n) = (\exp(rh) - 1)u(t_n)(1 - u(t_{n+1}))$$

$$\frac{u(t_{n+1}) - u(t_n)}{\frac{(\exp(rh) - 1)}{r}} = ru(t_n)(1 - u(t_{n+1}))$$

Fungsi Penyebut/ Denominator

V.S. Skema Euler

$$\frac{u(t_{n+1}) - u(t_n)}{h} = ru(t_n)(1 - u(t_n))$$

V.S. Modifikasi Skema Euler

$$\frac{u(t_{n+1}) - u(t_n)}{h} = ru(t_n)(1 - u(t_{n+1}))$$

Aproksimasi nonlokal

Beberapa Skema Eksak

$\frac{dy}{dt} = -\lambda y$	$\frac{y_{k+1}-y_k}{\left(1-e^{-\lambda\Delta t}\right)/\lambda} = -\lambda y_k$
$\frac{d^2y}{dt^2} + \omega^2 y = 0$	$\frac{y_{k+1}-2y_k+y_{k-1}}{\frac{4}{\omega^2}\sin^2\left(\frac{\Delta t\omega}{2}\right)} + \omega^2 y_k = 0$
$\frac{dy}{dt} = \lambda_1 y - \lambda_2 y^2$	$\frac{y_{k+1}-y_k}{\left(e^{\lambda_1\Delta t}-1\right)/\lambda_1} = \lambda_1 y_k - \lambda_2 y_{k+1} y_k$
$2\frac{dy}{dt} + y = \frac{1}{y}$	$\frac{2(y_{k+1}-y_k)}{(1-e^{-\Delta t})} + \frac{y_k^2}{\left(\frac{y_{k+1}+y_k}{2}\right)} = \frac{1}{\left(\frac{y_{k+1}+y_k}{2}\right)}$
$\frac{d^2y}{dt^2} = \lambda\frac{dy}{dt}$	$\frac{y_{k+1}-2y_k+y_{k-1}}{\left(\frac{e^{\lambda\Delta t}-1}{\lambda}\right)\Delta t} = \lambda\left(\frac{y_k-y_{k-1}}{\Delta t}\right)$
$u_t + u_x = u(1-u)$	$\frac{u_m^{k+1}-u_m^k}{e^{\Delta t}-1} + \frac{u_m^k-u_{m-1}^k}{e^{\Delta x}-1} = u_{m-1}^k\left(1-u_m^{k+1}\right)$ for $\Delta t = \Delta x$
$y_{tt} - y_{xx} = 0$	$y_m^{k+1} - 2y_m^k + y_m^{k-1} = y_{m+1}^k - 2y_m^k + y_{m-1}^k$
$\frac{d^2y}{dt^2} + 2\epsilon\frac{dy}{dt} + y = 0$	$\psi(\omega,\Delta t) = \frac{\epsilon e^{-\epsilon\Delta t}}{\sqrt{1-\epsilon^2}} + e^{-\epsilon\Delta t}\cos\left(\sqrt{1-\epsilon^2}\right)\Delta t, \phi(\epsilon,\Delta t)$ $= \frac{\epsilon e^{-\epsilon\Delta t}}{\sqrt{1-\epsilon^2}}\sin\left(\sqrt{1-\epsilon^2}\right)\Delta t$, $\frac{y_{k+1}-2y_k+y_{k-1}}{\phi^2} + 2\epsilon\left(\frac{y_k-\psi y_{k-1}}{\phi}\right) + \frac{2(1-\psi)y_k+(\phi^2+\psi^2-1)y_{k-1}}{\phi^2} = 0$
$\frac{\partial c}{\partial t} + P_{n-1}(t)\frac{\partial c}{\partial x} = \lambda c(1-c)$ $P_{n-1}(t) = \sum_{i=0}^{i=n-1} a_i t^i$	$\frac{C^{k+1}(x)-C^k(\tilde{x}^k)}{\left(e^{\lambda\Delta t}-1\right)/\lambda} = \lambda C^k(\tilde{x}^k)\left(1-C^{k+1}(x)\right)$, $\tilde{x}^k = x - [P_n((k+1)\Delta t) - P_n(k\Delta t)],\ P_n(t) = \int_0^t P_{n-1}(\tau)d\tau.$
$\frac{\partial c}{\partial t} + P_{n-1}(t)\frac{\partial c}{\partial x} = \lambda c$	$\frac{C^{k+1}(x)-C^k(\tilde{x}^k)}{\left(e^{\lambda\Delta t}-1\right)/\lambda} = \lambda C^k(\tilde{x}^k).$
$\frac{\partial c}{\partial t} + P_{n-1}(t)\frac{\partial c}{\partial x} = \mu + \lambda c$	$\frac{C^{k+1}(x)-C^k(\tilde{x}^k)}{\left(e^{\lambda\Delta t}-1\right)/\lambda} = \mu + \lambda C^k(\tilde{x}^k).$

Skema Eksak

Definisi

Suatu metode numerik untuk suatu persamaan diferensial disebut **skema eksak** apalabila persamaan diferensial dan persamaan beda (skema) tersebut mempunyai penyelesaian umum yang sama pada waktu diskret $t = t_n$.

Nonstandard Finite Difference Method

Nonstandard Finite Difference (NSFD) Method*

A numerical scheme for an initial value problem

$$\frac{d\vec{u}}{dt} = f(t, \vec{u}); \vec{u}(0) = \vec{u}_0$$

is called a NSFD method if at least one of the following conditions is satisfied [4,5]:

(i) LHS ➔ the generalization of forward difference scheme

$$\frac{d\vec{u}_n}{dt} \approx \frac{\vec{u}_{n+1} - \vec{u}_n}{\psi(h)}$$

The nonnegative denominator function has to satisfy $\psi(h) = h + \mathcal{O}(h^2), h = \Delta t$.

(ii) The approximation of $f(t, u)$ is nonlocal.

*R. Mickens, *Nonstandard finite difference models of differential equations*, World Scientific, 1994.

NonStandard Grünwald-Letnikov (NSGL) Approximation

$$x_{m+1} = \sum_{j=1}^{m+1} c_j^{\alpha} x_{m+1-j} + r_{m+1}^{\alpha} x_0 + s^{\alpha} \left[x_m \left(1 - \frac{x_{m+1}}{\gamma} \right) - \frac{(1-c')x_{m+1} y_m}{1 + \theta\xi + x_m} \right]$$

$$y_{m+1} = \sum_{j=1}^{m+1} c_j^{\alpha} y_{m+1-j} + r_{m+1}^{\alpha} y_0 + s^{\alpha} \left[\frac{\beta[(1-c')x_{m+1} + \xi] y_m}{1 + \theta\xi + x_m} - \delta y_{m+1} \right]$$

$$x_{m+1} = \frac{r_{m+1}^{\alpha} x_0 + s^{\alpha} x_m + \sum_{j=1}^{m+1} c_j^{\alpha} x_{m+1-j}}{1 + s^{\alpha} \left[\frac{x_m}{\gamma} + \frac{(1-c')y_m}{1 + \theta\xi + x_m} \right]}$$

$$y_{m+1} = \frac{r_{m+1}^{\alpha} y_0 + s^{\alpha} \left[\frac{\beta[(1-c')x_{m+1} + \xi] y_m}{1 + \theta\xi + x_m} \right] + \sum_{j=1}^{m+1} c_j^{\alpha} y_{m+1-j}}{1 + \delta s^{\alpha}}$$

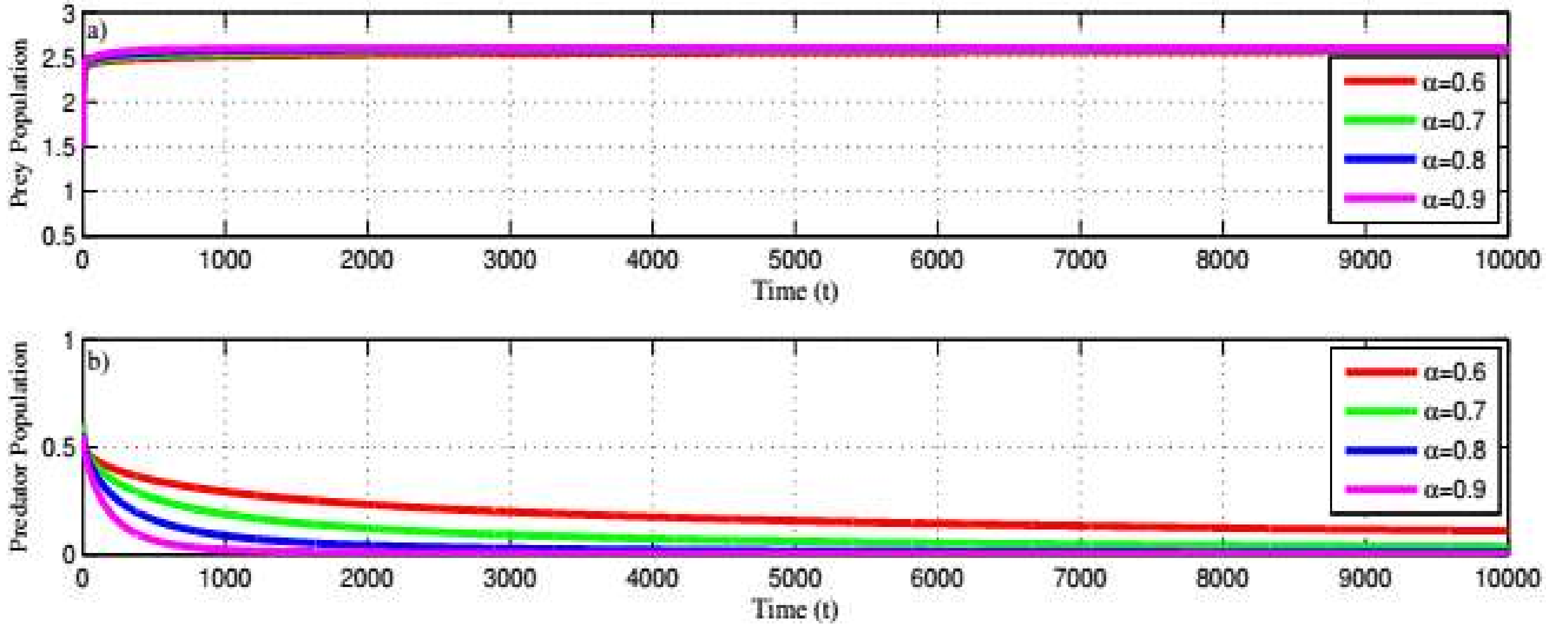


Figure 1: Numerical solution of system (2) with $\gamma = 2.6, c' = 0.6, \theta = 0.6, \xi = 0.2, \beta = 0.21, \delta = 0.08$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$: (a) Prey population $x(t)$, (b) Predator population $y(t)$.

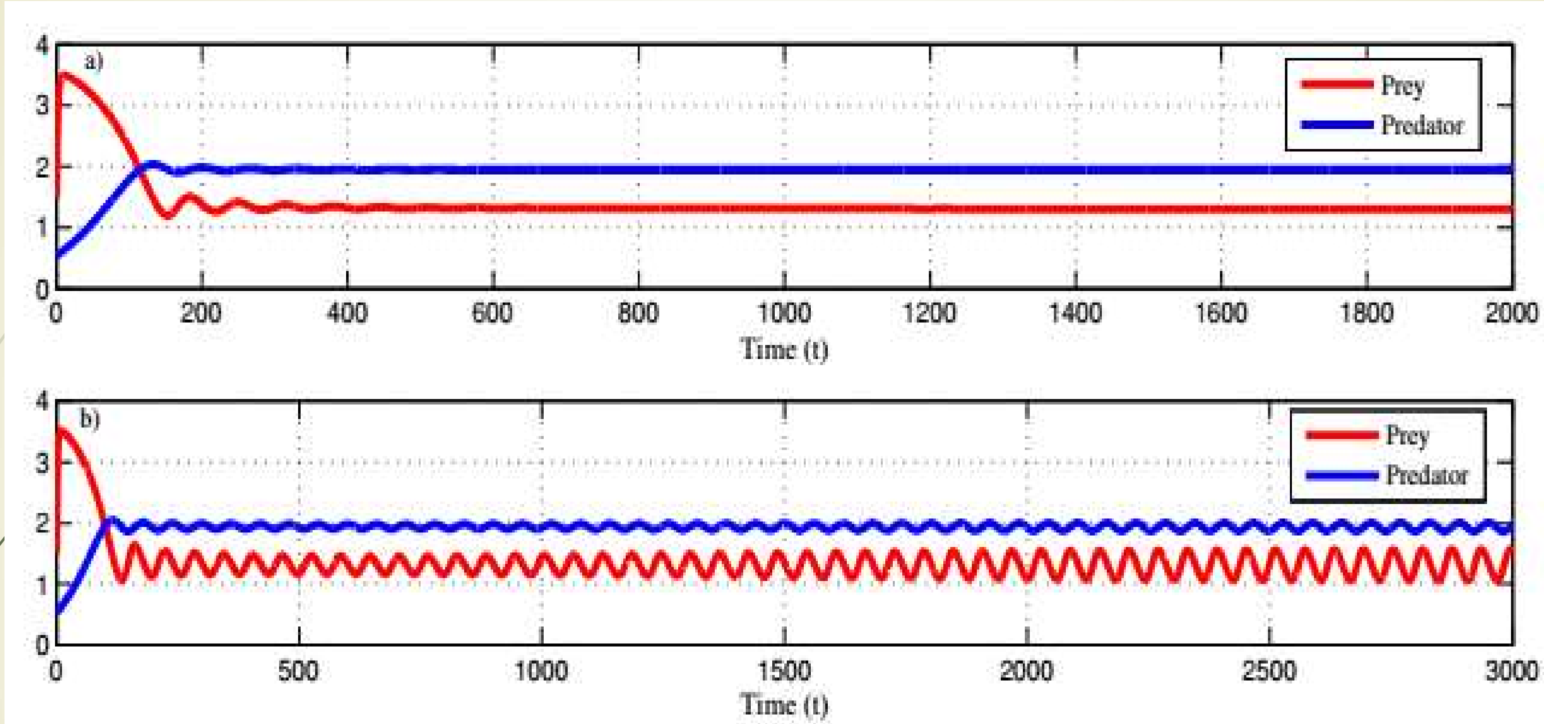


Figure 2: Numerical solution of system (2) with $\gamma = 4, c' = 0.16, \theta = 0.6, \xi = 0.2, \beta = 0.15$, $\delta = 0.08$ and (a) $\alpha = 0.88$, (b) $\alpha = 0.91$.

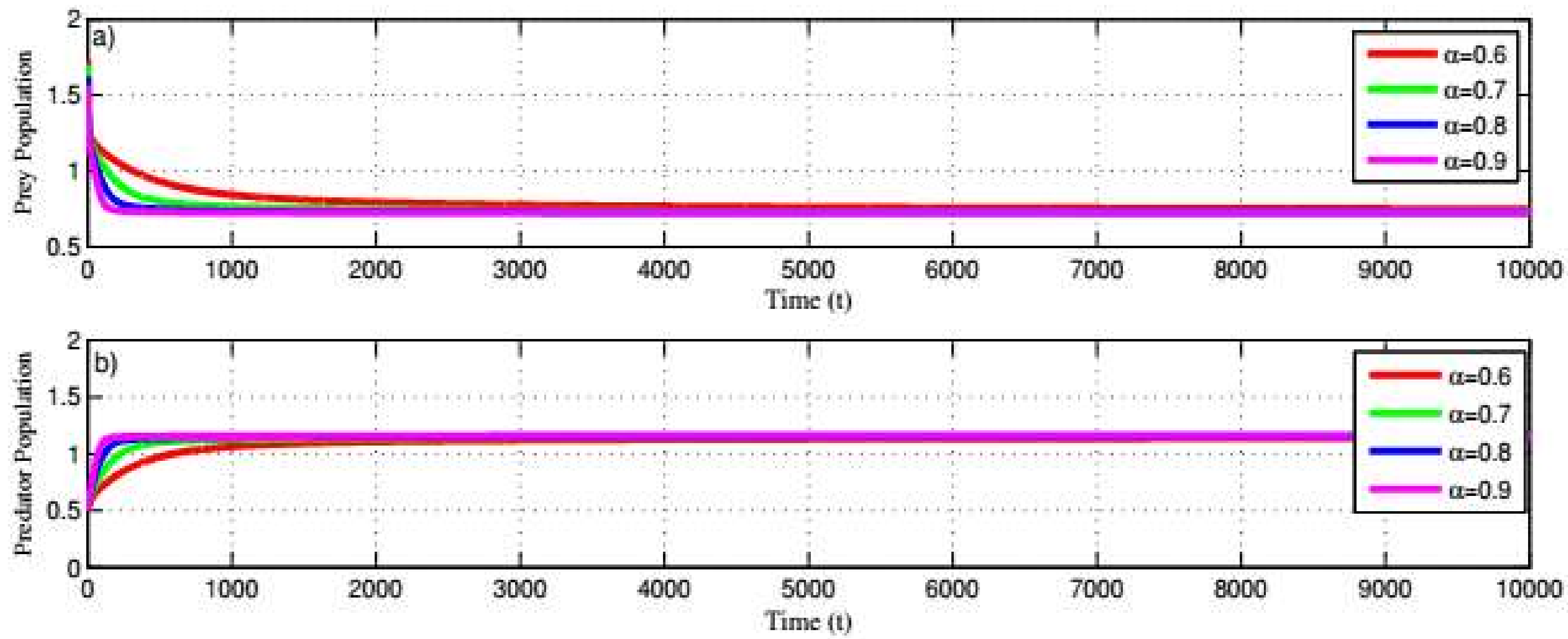


Figure 3: Numerical solution of system (2) with $\gamma = 1.5, c' = 0.22, \theta = 0.6, \xi = 0.1, \beta = 0.21, \delta = 0.08$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$: (a) Prey population $x(t)$ and (b) Predator population $y(t)$.

Concluding Remarks

1. Fractional-order model for predator-prey model has been discussed: non-negativity, boundedness, existence and uniqueness of solution have been presented.
2. Model has three equilibria: extinction point, predator extinction, coexistence point. Extinction point is unstable and others are conditionally stable.

References

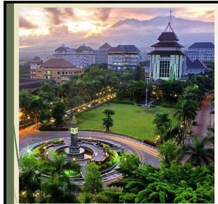
[1] Z. M. Odibat and N. T. Shawagfeh, Generalized Taylors formula, *Applied Mathematics and Computation* **186** (2007), 286-293, DOI: 10.1016/j.amc.2006.07.102.

[2] H. Li, L. Zhang, C. Hu, J. Yao-Lin and Z. Teng, Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge, *Journal of Applied Mathematics and Computing* **64** (2016), 435-449, DOI: 10.1007/s12190-016-1017-8.

[3] Y. Li, Y. Chen and I. Podlubny, Stability of fractional-order nonlinear dynamics: Lyapunov direct method and generalized Mittag-Leffler stability, *Computers & Mathematics with Applications* **59** (2010), 1810-1821, DOI: 10.1016/j.camwa.2009.08.019.

[4] C. V. De-León, Volterra-type lyapunov functions for fractional-order epidemic systems, *Communications in Nonlinear Science and Numerical Simulation* **24** (2015), 75-85, DOI: 10.1016/j.cnsns.2014.12.013.

[5] J. Huo, H. Zhao and L. Zhu, The effect of vaccines on backward bifurcation in a fractional order HIV model, *ISSN: 1468-1218 Nonlinear Analysis: Real World Applications* **26** (2015), 289-305, DOI: 10.1016/j.nonrwa.2015.05.014.



Model Predator-Prey Orde Fractional

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Convergence of Mittag-Leffler Function

Let $\alpha > 0$. The Mittag-Leffler function $E_\alpha(z)$, where $z = re^{i\theta}$, $r = |\lambda|$, $\theta = \arg(z)$, behaves as follows

1. $E_\alpha(re^{i\theta}) \to 0$ for $r \to \infty$ if $|\theta| > \frac{\alpha\pi}{2}$,
2. $E_\alpha(re^{i\theta})$ remains bounded for $r \to \infty$ if $|\theta| = \frac{\alpha\pi}{2}$,
3. $E_\alpha(re^{i\theta}) \to \infty$ for $r \to \infty$ if $|\theta| < \frac{\alpha\pi}{2}$,

Remark:

For classical $\alpha = 1$, Mittag-Leffler is reduced to exp(z). As, $|z| \to \infty$ then $\exp(z)$ (a) goes to zero as $|\arg(z)| > \pi/2$, (b) remains bounded if $|\arg(z)| = \pi/2$, (c) grows without bound if $|\arg(z)| < \pi/2$.

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Introduction to Fractional Calculus*

I. Special functions

Gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$
$$\Gamma(1) = \Gamma(2) = 1$$
$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(n) = (n-1)!;\ n \in N$$
$$\Gamma(1/2) = \sqrt{\pi}$$
$$\Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^n}(2n-1)!;\ n \in N$$

$$\exp(x) = \sum_{j=0}^{\infty}\frac{x^j}{j!} = \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+1)}$$

$\alpha = 1$ ↑ ↓ Mittag-Leffler function:

$$E_\alpha(x) = \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\alpha j+1)};\ \alpha > 0,\ \alpha \in R,\ x \in C$$

Two-parameter generalization:

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\alpha j+\beta)};\ \alpha, \beta > 0,\ \alpha, \beta \in R,\ x \in C$$

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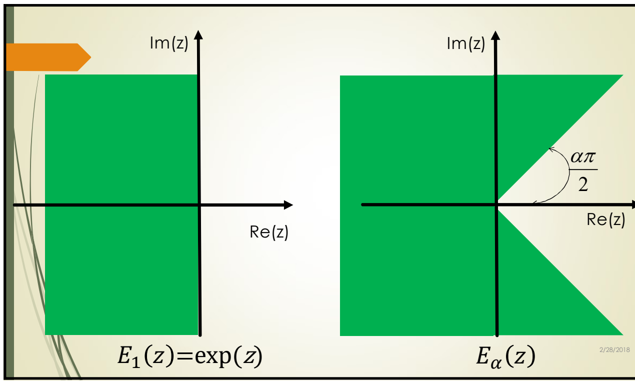
I. Petras, *Fractional-order nonlinear systems: Modeling, Analysis and Simulation*, Springer, 2011.



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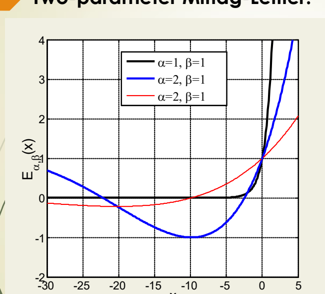
Two-parameter Mittag-Leffler:



$$E_{\alpha,1}(x) = E_\alpha(x)$$
$$E_{1,1}(x) = \exp(x)$$
$$E_{2,1}(x^2) = \sum_{j=0}^{\infty}\frac{x^{2j}}{\Gamma(2j+1)} = \sum_{j=0}^{\infty}\frac{x^{2j}}{(2j)!} = \cosh(x)$$
$$E_{2,2}(x^2) = \sum_{j=0}^{\infty}\frac{x^{2j}}{\Gamma(2j+2)} = \frac{1}{x}\sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!} = \frac{\sinh(x)}{x}$$

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Introduction to Fractional Calculus*

Definition 1. Let $\alpha \in (0,\infty)$, the operator J^α defined by

$$J^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau, \quad (t \in [0,a])$$

is called **the Riemann-Liouville fractional integral operator of order α**, where $J^0 = Id$ is the identity operator.

Definition 2. Let $\alpha \in (0,\infty)$ and $m = \lceil\alpha\rceil$, where $\lceil t\rceil := \min\{k \in \mathbb{Z}: k \geq \alpha\}$, the operator D^α defined for u by

$$D^\alpha u(t) := \frac{d^m}{dt^m}J^{m-\alpha}u(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}u(\tau)d\tau,$$

is called **the Riemann-Liouville fractional differential operator of order α**.

I. Petras, Fractional-order nonlinear systems: Modeling, Analysis and Simulation, Springer, 2011.

Introduction to Fractional Calculus

Definition 3. Assume that $\alpha \geq 0$ and u is such that $J^{m-\alpha}u^{(m)}$ exists, where $m = \lceil \alpha \rceil$. **the Caputo fractional differential operator of order α** is defined by

$$D_*^\alpha u(t) := J^{m-\alpha}u^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}u^{(m)}(\tau)d\tau$$

The Reimann-Liouville vs. Caputo fractional differential operator:

$$D_*^\alpha u(t) = D^\alpha u(t) - \sum_{v=0}^{m-1} r_v^\alpha(t)u^{(v)}(0);\ r_v^\alpha(t) = \frac{t^{v-\alpha}}{\Gamma(v+1-\alpha)}$$

For the case $m = 1$ or $0 < \alpha < 1$, then "correction term reads"

$$D_*^\alpha u(t) = D^\alpha u(t) - r_0^\alpha(t)y_0$$

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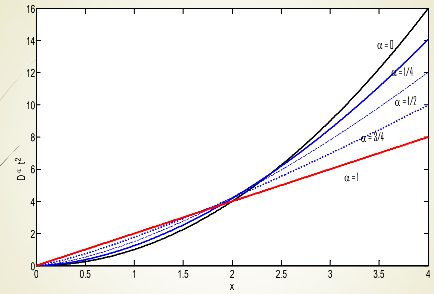

$$D_*^\alpha t^2 = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha},\ 0<\alpha<1$$

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Example:

$a = 0,\ \alpha = 1/2\ (m=1), u(t) = t$

$$D_*^{1/2}t = \frac{1}{\Gamma(1/2)}\int_0^t \frac{1}{(t-\tau)^{1/2}}d\tau$$
$$= -\frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}}d(t-\tau)$$
$$= -\frac{1}{\sqrt{\pi}}\int_{\sqrt{t}}^0 \frac{1}{u}du^2$$
$$= \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{t}} 2du = \frac{2\sqrt{t}}{\sqrt{\pi}}$$

	$D_*^\alpha f(t)$	$D_*^{1/3}f(t)$	$D_*^{1/2}f(t)$
$f(t) = const$	0	0	0
$f(t) = t$	$\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}$	$1.1077\,t^{2/3}$	$1.1284\,t^{1/2}$
$f(t) = t^2$	$\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}$	$1.3293\,t^{5/3}$	$1.5045\,t^{3/2}$
$f(t) = t^3$	$\frac{6}{\Gamma(4-\alpha)}t^{3-\alpha}$	$1.4954\,t^{8/3}$	$1.8054\,t^{5/2}$
$f(t) = t^4$	$\frac{24}{\Gamma(5-\alpha)}t^{4-\alpha}$	$1.6314\,t^{11/3}$	$2.0633\,t^{7/2}$
$f(t) = t^5$	$\frac{120}{\Gamma(6-\alpha)}t^{5-\alpha}$	$1.7479\,t^{14/3}$	$2.2926\,t^{9/2}$
$f(t) = t^{1/2}$	$\frac{\sqrt{\pi}}{2\,\Gamma(3/2-\alpha)}t^{1/2-\alpha}$	$0.9553\,t^{1/6}$	0.8862
$f(t) = t^{3/2}$	$\frac{3\sqrt{\pi}}{4\,\Gamma(5/2-\alpha)}t^{3/2-\alpha}$	$1.2282\,t^{7/6}$	$1.3292\,t$
$f(t) = e^t$	$t^{n-\alpha}E_{1,n-\alpha+1}(t)$	$t^{2/3}E_{1,5/3}(t)$	$t^{1/2}E_{1,3/2}(t)$

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Fractional Initial Value Problem

The solution of linear initial value problem (IVP):

$$D_*^\alpha u(t) = \lambda u(t);\ t>0,\ m-1<\alpha<m,$$
$$u^{(k)}(0) = b_k,\ b_k \in R,\ k = 0,1,\ldots,m-1$$

is given by

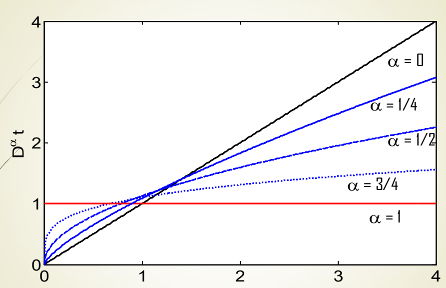
$$u(t) = \sum_{k=0}^{m-1} b_k t^k E_{\alpha,k+1}(\lambda t^\alpha)$$

where $E_{\alpha,\beta}(x)$ is the two-parameter function of Mittag-Leffler type.

Proof: use Laplace transform, see Podlubny, Subsection 1.4

2/28/2018

I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999



$$D_*^\alpha t = \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha},\ 0<\alpha<1$$

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Fractional Initial Value Problem

$$D_*^\alpha u(t) = \lambda u(t);\ t>0,\ 0<\alpha<1,$$
$$u(0) = b_0,\ b_0 \in R$$

$\Rightarrow$

$$u(t) = b_0 E_{\alpha,1}(\lambda t^\alpha) = b_0 E_\alpha(\lambda t^\alpha)$$
$$= b_0 \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^j}{\Gamma(\alpha k+1)}$$

- Suppose $\alpha > 0$ and $\lambda = re^{i\theta}$ where $r = |\lambda|, \theta = arg(\lambda)$.

1. $E_\alpha(re^{i\theta}) \to 0$ as $r \to \infty$ if $arg(\lambda) = |\theta| > \frac{\alpha\pi}{2}$,
2. $E_\alpha(re^{i\theta})$ remains bounded as $r \to \infty$ if $arg(\lambda) = |\theta| = \frac{\alpha\pi}{2}$,
3. $E_\alpha(re^{i\theta}) \to \infty$ as $r \to \infty$ if $arg(\lambda) = |\theta| < \frac{\alpha\pi}{2}$

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Fractional (system) Initial Value Problem

$$D_*^{\alpha}\vec{u}(t) = A\vec{u}(t), t > 0, 0 < \alpha, \vec{u} \in \mathbb{R}^n, A_{n\times n}$$
$$\vec{u}(0) = \vec{b}_0, \vec{b}_0 \in \mathbb{R}^n$$

$B^{-1}AB = C$

$AB = BC$

$A = BCB^{-1}$

$$D_*^{\alpha}\vec{u}(t) = A\vec{u}(t) = (BCB^{-1})\vec{u}(t)$$
$$D_*^{\alpha}(B^{-1}\vec{u}(t)) = C(B^{-1}\vec{u}(t))$$
$$D_*^{\alpha}(\vec{v}(t)) = C(\vec{v}(t)); \vec{v}(t) = B^{-1}\vec{u}(t)$$

Mathematical Model (1)

Ghosh et al (2017)*:
$$\frac{dx}{dt} = x\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')xy}{1+\theta\xi + x}$$
$$\frac{dy}{dt} = \frac{\beta[(1-c')x + \xi]y}{1+\theta\xi+x} - \delta y$$

The growth rate depends instantly on the current state !

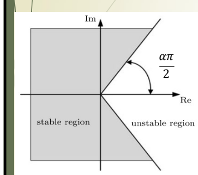
How to include the previous history (memory effects)?

⬇

Fractional-order derivative

*Ghosh, J., B. Sahoo, and S. Poria. 2017, Prey-predator dynamics with prey refuge providing additional food to predator, *Chaos, Solitons and Fractals*, 96: 110–119.

Autonomous Nonlinear Fractional System



Theorem 1*. Consider the following autonomous nonlinear fractional-order system

$$D_*^{\alpha}\vec{u} = \vec{f}(\vec{u}); \quad \vec{u}(0) = \vec{u}_0; \quad 0 < \alpha < 1.$$

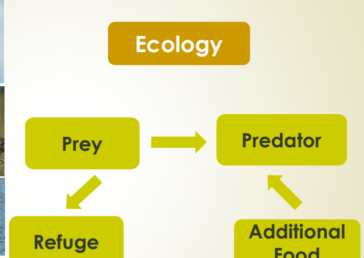
The equilibrium points of the above system are solutions to the equation $\vec{f}(\vec{u}) = 0$. An equilibrium is locally asymptotically if all eigenvalues λ_j of the Jacobian matrix $J = \frac{\partial \vec{f}}{\partial \vec{u}}$ at the equilibrium satisfy $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$.

*D. Matignon, Stability results for fractional differential equations with application to control processing, in: *Appl. Computational Eng. Sys.* 2, France, 1996, pp. 963-968

I. Petras, *Fractional-order nonlinear systems: Modeling, Analysis and Simulation*, Springer, 2011.

Mathematical Model (2)

$$D_*^{\alpha}x = x\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')xy}{1+\theta\xi + x}$$
$$D_*^{\alpha}y = \frac{\beta[(1-c')x + \xi]y}{1+\theta\xi+x} - \delta y$$



Non-negativity of solution

Lemma 1. [1]
Let $0 < \alpha \leq 1, u(t) \in C[a,b]$ and $D_*^{\alpha}u(t) \in C[a,b]$. Then the following statements hold:

1. If $D_*^{\alpha}u(t) \geq 0, \forall t \in (a,b)$, then $u(t)$ is a non-decreasing function for all $t \in [a,b]$
2. If $D_*^{\alpha}u(t) \leq 0, \forall t \in (a,b)$, then $u(t)$ is a non-increasing function for all $t \in [a,b]$

Theorem 2.
All solution of model (2) with initial condition $x(0) \geq 0, y(0) \geq 0$ are non-negative.

Proof:
We will prove that $x(t) \geq 0, y(t) \geq 0$ for all $t \geq 0$. Suppose this is not true, then there is a constant $t_1 > 0$ such that $x(t) > 0$ for $0 \leq t < t_1$; $x(t_1) = 0$; and $x(t_1^+) < 0$. Substituting $x(t_1) = 0$ into model (2) gives

$$D_*^{\alpha}x(t_1)\Big|_{x(t_1)=0} = 0.$$

Since $D_*^{\alpha}x(t_1) = 0$, Lemma 1 says that $x(t_1^+) = 0$ which contradicts with $x(t_1^+) < 0$. Hence $x(t) \geq 0$ for all $t \geq 0$. Similar argument can be used to prove that $y(t) \geq 0$ for all $t \geq 0$.

Boundedness of solution

Lemma 3.[2]
Let $u(t)$ be a continuous function on $[t_0, +\infty)$ and satisfying

$$D_*^\alpha u(t) \leq -\lambda u(t) + \mu;\ u(t_0) = u_0$$

where $0 < \alpha < 1, (\lambda, \mu) \in \mathbb{R}^2$ and $\lambda \neq 0$, and $t_0 \geq 0$ is the initial time.
Then $u(t) \leq \left(u_0 - \frac{\mu}{\lambda}\right) E_\alpha[-\lambda(t-t_0)^\alpha] + \frac{\mu}{\lambda}$.

Theorem 4.
Solution of model (2) with initial condition $x(0) \geq 0, y(0) \geq 0$ is uniformly bounded.

Proof:
First define $w(t) = x(t) + \frac{1}{\beta} y(t)$ and $\sigma(x) = 1 + \theta\xi + x$ such that

$$D_*^\alpha w(t) + (\beta\xi - \delta) w(t) = x - \frac{1}{\gamma} x^2 + (\beta\xi - \delta)x + \left(\frac{1-\sigma(x)}{\sigma(x)}\right)\xi y$$
$$\leq -\frac{1}{\gamma}\left(x - \frac{\gamma(1+\beta\xi-\delta)}{2}\right)^2 + \frac{\gamma(1+\beta\xi-\ \ )^2}{4}$$
$$\leq \frac{\gamma(1+\beta\xi-\delta)^2}{4}$$

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Equilibrium Point and Stability

$$D_*^\alpha x(t) = D_*^\alpha y(t) = 0 \quad \Rightarrow \quad x\left[\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')y}{1+\theta\xi+x}\right] = 0$$
$$y\left[\frac{\beta[(1-c')x+\xi]}{1+\theta\xi+x} - \delta\right] = 0.$$

1. Extinction point: $E_0 = (0,0)$
2. Predator extinction point: $E_1 = (\gamma, 0)$
3. Coexistence point: $E_* = (x^*, y^*)$ where $x^* = \frac{\delta + (\delta\theta - \beta)\xi}{\beta(1-c') - \delta}$ and $y^* = \left(1 - \frac{x^*}{\gamma}\right)\left(\frac{1+\theta\xi+x^*}{(1-c')}\right)$. E_* exists if $x^* < \gamma$ and $\beta(1-c') < \delta < \frac{\beta\xi}{1+\theta\xi}$.

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Boundedness of solution

Using Lemma 3, we have

$$w(t) \leq \left(w(0) - \frac{\gamma(1+\beta\xi-\delta)^2}{4(\beta\xi-\delta)}\right) E_\alpha[-(\beta\xi-\delta)t^\alpha] + \frac{\gamma(1+\beta\xi-\delta)^2}{4(\beta\xi-\delta)}$$

For $t \to \infty$, we have $w(t) \to \frac{\gamma(1+\beta\xi-\delta)^2}{4(\beta\xi-\delta)}$.

Hence, all solutions which start from $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+\}$ are confined to the region

$$\Gamma = \left\{(x,y) \in \Omega \middle| x + \frac{1}{\beta} y \leq \frac{\gamma(1+\beta\xi-\delta)^2}{4(\beta\xi-\delta)} + \epsilon, \epsilon > 0\right\}.$$

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Local Stability

Jacobian matrix at equilibrium (x^*, y^*)

$$J(x^*, y^*) = \begin{bmatrix} 1 - \frac{2x^*}{\gamma} - \frac{(1-c')(1+\theta\xi)y^*}{(1+\theta\xi+x^*)^2} & \frac{-(1-c')x^*}{1+\theta\xi+x^*} \\ \frac{\beta y^*\left[(1-c')(1+\theta\xi)-\xi\right]}{(1+\theta\xi+x^*)^2} & \frac{\beta\left[(1-c')x^*+\xi\right]}{1+\theta\xi+x^*} - \delta \end{bmatrix}$$

Theorem 8.
1. Equilibrium point E_0 is unstable
2. Equilibrium point E_1 is asymptotically stable if $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+\gamma} < \delta$

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Lemma 5. [3] **Existence and Uniqueness of solution**
Consider a fractional-order differential system

$$D_*^\alpha u(t) = f(t, u(t)), t > 0$$

with initial condition $x(0) \geq 0, y(0) \geq 0$ and $0 < \alpha < 1$, $f: (0,\infty) \times \Psi \to \mathbb{R}^2, \Psi \subseteq \mathbb{R}^2$. If $u(t)$ satisfies the Lipschitz condition w.r.t u, then there exists a unique solution of the above system on : $(0,\infty) \times \Psi$.

Theorema 6.
Consider model (2) with initial condition $x(0) \geq 0, y(0) \geq 0$ and $0 < \alpha < 1$, $f: (0,\infty) \times \Omega_M \to \mathbb{R}^2$, where $\Omega_M = \{(x,y) \in \mathbb{R}_+^2 | max\{|x|,|y|\} \leq M\}$ for sufficiently large M. This IVP has a unique solution.

Proof:
Consider $H(X) = (H_1(X), H_2(X))$ with

$$H_1(X) = x\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')xy}{1+\theta\xi+x};\ H_2(X) = \frac{\beta[(1-c')x+\xi]y}{1+\theta\xi+x} - \delta y$$

For any , $X = (x,y), \bar{X} = (\bar{x},\bar{y}), X, \bar{X} \in \Omega_M$, we can show that

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$$\|H(X) - H(\bar{X})\| \leq L\|X - \bar{X}\|, L \equiv L(\gamma, c', \theta, \beta, M)$$

At $E^* = (x^*, y^*)$, the Jacobian matrix has a characteristics equation

$$\lambda^2 - a_1\lambda + a_2 = 0,$$
$$a_1 = \left[\frac{x^*}{\gamma}\left(\frac{\gamma - x^*}{1+\theta\xi+x^*} - 1\right)\right],\ a_2 = \frac{\beta(1-c')\left[(1-c')(1+\theta\xi)-\xi\right]x^*y^*}{(1+\theta\xi+x^*)^3}$$
$$\lambda_{1,2} = \frac{1}{2}\left(a_1 \pm \sqrt{D}\right),\ D = (a_1)^2 - 4a_2.$$

Theorem 9.
Equilibrium E^* is asymptotically stable if one of the following mutually exclusive conditions holds:
(i) $a_1 < 0; a_2 > 0$ and $D \geq 0$
(ii) $D < 0$ and $\left|\sqrt{D}/a_1\right| > \tan(\frac{\alpha\pi}{2})$

Proof.
(i) If $a_1 < 0$; $a_2 > 0$ and $D \geq 0$ then $\lambda_{1,2} < 0$, hence $\arg(\lambda_{1,2}) = \pi > \frac{\alpha\pi}{2}$ and the result follows.
(ii) Suppose $D < 0$. If λ is an eigenvalue, then $\bar{\lambda}$ is also an eigenvalue. Using $\left|\sqrt{D}/a_1\right| > \tan(\frac{\alpha\pi}{2})$, we have that $\left|\frac{\lambda-\bar{\lambda}}{\lambda+\bar{\lambda}}\right| = \left|\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right| = |\arg(\lambda)| = \left|\sqrt{D}/a_1\right| > \tan(\frac{\alpha\pi}{2})$. Therefore the stability of E^* follows. □

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Global Stability

Lemma 10. [4]
Let $x(t) \in \mathbb{R}_+$ be continuous and derivable function. Then, for any $t > t_0$ and $\alpha \in (0,1)$:

$$D_*^\alpha \left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*} \right] \leq \left(1 - \frac{x^*}{x(t)} \right) D_*^\alpha x(t), x^* \in \mathbb{R}_+$$

Lemma 11. ***Generalized Lasalle Invariance Principle* [5]**
Suppose D is a bounded closed set and every solution of

$$D_*^\alpha x(t) = f(x)$$

starts from a point and remains in D for all time. *If* $\exists V(x): D \to \mathbb{R}$ with continuous first partial derivatives satisfies

$$D_*^\alpha V|_{D_*^\alpha x(t) = f(x)} \leq 0.$$

Let $E = \{ x | D_*^\alpha V|_{D_*^\alpha x(t) = f(x)} = 0 \}$ and M be the largest invariant set of E. Then every solution $x(t)$ originating in D tends to M as $t \to \infty$.

Integer-Order Derivative

$$\frac{du}{dt} = u'(t) = \lim_{h \to 0} \left(\frac{u(t) - u(t-h)}{h} \right)$$

$$\frac{d^2u}{dt^2} = u''(t) = \lim_{h \to 0} \left(\frac{u'^{(t)} - u'^{(t-h)}}{h} \right) = \lim_{h \to 0} \frac{1}{h} \left(\frac{u(t) - u(t-h)}{h} - \frac{u(t-h) - u(t-2h)}{h} \right) = \lim_{h \to 0} \frac{u(t) - 2u(t-h) + u(t-2h)}{h^2}$$

$$\frac{d^3u}{dt^3} = u'''(t) = \lim_{h \to 0} \frac{u(t) - 3u(t-h) + 3u(t-2h) - u(t-3h)}{h^3}$$

$$\frac{d^nu}{dt^n} = u^n(t) = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} u(t - jh), \qquad \binom{n}{j} = \frac{n(n-1)(n-2)\ldots(n-j+1)}{j!} = \frac{n!}{j!\,(n-j)!} = \frac{\Gamma(n+1)}{\Gamma(j+1)\Gamma(n-j+1)}$$

Theorem 12.
E_1 is globally asymptotically stable if $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+x} \leq \delta$.

Proof:
1. Define a Lyapunov function $U(x,y) = \left(x - \gamma - \gamma \ln \frac{x}{\gamma} + \frac{y}{\beta} \right)$, then

$$D_*^\alpha U(x,t) \leq \frac{x-\gamma}{x} D_*^\alpha x(t) + \frac{1}{\beta} D_*^\alpha y(t) = -\frac{1}{\gamma}(x-\gamma)^2 + \left[\frac{(1-c')\gamma + \xi}{1+\theta\xi+x} - \frac{\delta}{\beta} \right] y$$

If $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+x} \leq \delta$ then $D_*^\alpha U(x,t) \leq 0$ for all $(x,t) \in \mathbb{R}_+^2$. Furthermore $D_*^\alpha U(x,t) = 0$ implies that $x = \gamma$ and $y = 0$. Hence, the only invariant set on which $D_*^\alpha U(x,t) = 0$ is singleton $\{E_1\}$.

Lasalle Invariance Principle ➡ E_1 is globally stable

Grünwald-Letnikov Approximation*

Using the notation of finite difference of an equidistant grid in $[0, t_{n+1}]$, $t_{n+1} \in R$, Grünwald-Letnikov defines

$$D^\alpha u(t_{n+1}) = \lim_{h \to 0} \frac{1}{h^\alpha} \Delta_h^\alpha u(t_{n+1}).$$

$$\frac{1}{h^\alpha} \Delta_h^\alpha u(t_{n+1}) = \frac{1}{h^\alpha} \left(u(t_{n+1}) - \sum_{j=1}^{n+1} c_j^\alpha \, u(t_{n+1-j}) \right)$$

$$c_j^\alpha = (-1)^{j-1} \binom{\alpha}{j} = (-1)^{j-1} \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}$$

$$(1-z)^\alpha = 1 - \sum_{j=1}^{\infty} c_j^\alpha z^j; \qquad c_j^\alpha = \left(1 - \frac{\alpha+1}{j} \right) c_{j-1}^\alpha, \quad c_1^\alpha = \alpha$$

$$0 < c_{n+1}^\alpha < c_n^\alpha < \cdots < c_1^\alpha = \alpha$$

*R. Scherer, S.L. Kalla, Y. Tang, J. Huang, The Grünwald-Letnikov method for fractional differential equations, *Comput. Math. Appl.* **62**, 2011, pp. 902-917

Theorem 13.
E^* is globally asymptotically stable in the region $\Omega = \left\{ (x,y): \frac{y}{y^*} > \frac{x}{x^*} > 1 \right\}$.

Proof:
1. Define a Lyapunov function

$$V(x,y) = \left(x - x^* - x^* \ln \frac{x}{x^*} \right) + \frac{1}{\beta} \left(y - y^* - y^* \ln \frac{y}{y^*} \right)$$

then

$$D_*^\alpha V(x,t) \leq \frac{x - x^*}{x} D_*^\alpha x(t) + \frac{1}{\beta} \left(\frac{y - y^*}{y} \right) D_*^\alpha y(t)$$

$$= -\frac{1}{\gamma}(x - x^*)^2 - \frac{\xi(x-x^*)(y-y^*)}{(1+\theta\xi+x)(1+\theta\xi+x^*)} - \frac{(1-c')(x-x^*)(x^*y - xy^*)}{(1+\theta\xi+x)(1+\theta\xi+x^*)}$$

For any $(x,y) \in \Omega$ we have $D_*^\alpha V(x,t) \leq 0$. Furthermore $D_*^\alpha V(x,t) = 0$ implies that $x = x^*$ and $y = y^*$.
Hence, singleton $\{E^*\}$ is the only invariant set on which $D_*^\alpha U(x,t) = 0$.

➡ E^* is globally asymptotically stable

Grünwald-Letnikov Approximation*

Hence

$$D_*^\alpha u(t) \approx \frac{1}{h^\alpha} \Delta_h^\alpha u(t) - r_0^\alpha(t) y_0; \;\; r_0^\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}; 0 < \alpha < 1$$

*R. Scherer, S.L. Kalla, Y. Tang, J. Huang, The Grünwald-Letnikov method for fractional differential equations, *Comput. Math. Appl.* **62**, 2011, pp. 902-917

Persamaan Eksponensial: Skema Eksak

$$\frac{du(t)}{dt} = ru(t);\ u(0) = u_0;\ r \neq 0 \qquad u(t) = u_0 \exp(rt)$$

$$u(t_{n+1}) - u(t_n) = u_0 \exp(r(t_n + h)) - u_0 \exp(r(t_n))$$
$$= u(t_n)(\exp(rh) - 1)$$

$$\frac{u_{n+1} - u_n}{\frac{(\exp(rh) - 1)}{r}} = ru_n \quad \text{V.S. Skema Euler} \quad \frac{u_{n+1} - u_n}{h} = ru_n$$

Fungsi Penyebut/ Denominator

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Beberapa Skema Eksak

$\frac{dy}{dt} = -\lambda y$	$\frac{y_{k+1} - y_k}{(1 - e^{-\lambda \Delta t})/\lambda} = -\lambda y_k$
$\frac{d^2y}{dt^2} + \omega^2 y = 0$	$\frac{y_{k+1} - 2y_k + y_{k-1}}{\frac{4}{\omega^2}\sin^2\left(\frac{\Delta t \omega}{2}\right)} + \omega^2 y_k = 0$
$\frac{dy}{dt} = \lambda_1 y - \lambda_2 y^2$	$\frac{y_{k+1} - y_k}{(e^{\lambda_1 \Delta t} - 1)/\lambda_1} = \lambda_1 y_k - \lambda_2 y_{k+1} y_k$
$2\frac{dy}{dt} + y = \frac{1}{y}$	$\frac{2(y_{k+1} - y_k)}{(1 - e^{-\Delta t})} + \frac{y_k^2}{\left(\frac{y_{k+1} + y_k}{2}\right)} = \frac{1}{\left(\frac{y_{k+1} + y_k}{2}\right)}$
$\frac{d^2y}{dt^2} = \lambda \frac{dy}{dt}$	$\frac{y_{k+1} - 2y_k + y_{k-1}}{\left(\frac{e^{\lambda \Delta t} - 1}{\lambda}\right)\Delta t} = \lambda\left(\frac{y_k - y_{k-1}}{\Delta t}\right)$
$u_t + u_x = u(1 - u)$	$\frac{u_m^{k+1} - u_m^k}{e^{\Delta t} - 1} + \frac{u_m^k - u_{m-1}^k}{e^{\Delta x} - 1} = u_{m-1}^k\left(1 - u_m^{k+1}\right)$ for $\Delta t = \Delta x$
$y_{tt} - y_{xx} = 0$	$y_m^{k+1} - 2y_m^k + y_m^{k-1} = y_{m+1}^k - 2y_m^k + y_{m-1}^k$
$\frac{d^2y}{dt^2} + 2\epsilon\frac{dy}{dt} + y = 0$	$\psi(\omega, \Delta t) = \frac{\epsilon e^{-\epsilon \Delta t}}{\sqrt{1-\epsilon^2}} + e^{-\epsilon \Delta t}\cos\left(\sqrt{1-\epsilon^2}\right)\Delta t, \phi(\epsilon, \Delta t)$ $= \frac{\epsilon e^{-\epsilon \Delta t}}{\sqrt{1-\epsilon^2}}\sin\left(\sqrt{1-\epsilon^2}\right)\Delta t$, $\frac{y_{k+1} - 2y_k + y_{k-1}}{\phi^2} + 2\epsilon\left(\frac{y_k - \psi y_{k-1}}{\phi}\right) + \frac{2(1-\psi)y_k + (\phi^2 + \psi^2 - 1)y_{k-1}}{\phi^2} = 0$
$\frac{\partial c}{\partial t} + P_{n-1}(t)\frac{\partial c}{\partial x} = \lambda c(1 - c)$ $P_{n-1}(t) = \sum_{i=0}^{i=n-1} a_i t^i$	$\frac{C^{k+1}(x) - C^k(\tilde{x}^k)}{(e^{\lambda \Delta t} - 1)/\lambda} = \lambda C^k(\tilde{x}^k)\left(1 - C^{k+1}(x)\right)$, $\tilde{x}^k = x - [P_n((k+1)\Delta t) - P_n(k\Delta t)],\ P_n(t) = \int_0^t P_{n-1}(\tau)d\tau.$
$\frac{\partial c}{\partial t} + P_{n-1}(t)\frac{\partial c}{\partial x} = \lambda c$	$\frac{C^{k+1}(x) - C^k(\tilde{x}^k)}{(e^{\lambda \Delta t} - 1)/\lambda} = \lambda C^k(\tilde{x}^k).$
$\frac{\partial c}{\partial t} + P_{n-1}(t)\frac{\partial c}{\partial x} = \mu + \lambda c$	$\frac{C^{k+1}(x) - C^k(\tilde{x}^k)}{(e^{\lambda \Delta t} - 1)/\lambda} = \mu + \lambda C^k(\tilde{x}^k).$

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Persamaan Logistik: Skema Eksak

$$\frac{du(t)}{dt} = ru(t)(1 - u(t));\ u(0) = u_0;\ r > 0 \qquad u(t) = \frac{u_0 \exp(rt)}{1 + u_0(\exp(rt) - 1)}$$

$$u(t_{n+1}) = \frac{u_0 \exp(rt_{n+1})}{1 + u_0(\exp(rt_{n+1}) - 1)} = \frac{u_0 \exp(rt_n)\exp(rh)}{1 + u_0(\exp(rt)\exp(rh) - 1)}$$
$$= \frac{u_0 \exp(rt_n)\exp(rh)}{1 + u_0(\exp(rt_n) - 1) + u_0 \exp(rt_n)(\exp(rh) - 1)}$$
$$= \frac{\exp(rh)\left(\frac{u_0 \exp(rt_n)}{1 + u_0(\exp(rt_n) - 1)}\right)}{1 + \left(\frac{u_0 \exp(rt_n)}{1 + u_0(\exp(rt_n) - 1)}\right)(\exp(rh) - 1)} = \frac{u(t_n)\exp(rh)}{1 + u(t_n)(\exp(rh) - 1)}$$

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Skema Eksak

Definisi

Suatu metode numerik untuk suatu persamaan diferensial disebut **skema eksak** apalabila persamaan diferensial dan persamaan beda (skema) tersebut mempunyai penyelesaian umum yang sama pada waktu diskret $t = t_n$.

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Persamaan Logistik: Skema Eksak

$$u(t_{n+1}) = \frac{u(t_n)\exp(rh)}{1 + u(t_n)(\exp(rh) - 1)}$$

$$u(t_{n+1}) = u(t_n)\exp(rh) - u(t_n)u(t_{n+1})\exp(rh) + u(t_n)u(t_{n+1})$$
$$u(t_{n+1}) - u(t_n) = (\exp(rh) - 1)u(t_n)(1 - u(t_{n+1}))$$

Aproksimasi nonlokal

$$\frac{u(t_{n+1}) - u(t_n)}{\frac{(\exp(rh) - 1)}{r}} = ru(t_n)(1 - u(t_{n+1}))$$

V.S. Skema Euler
$$\frac{u(t_{n+1}) - u(t_n)}{h} = ru(t_n)(1 - u(t_n))$$

V.S. Modifikasi Skema Euler
$$\frac{u(t_{n+1}) - u(t_n)}{h} = ru(t_n)(1 - u(t_{n+1}))$$

Fungsi Penyebut/ Denominator

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Nonstandard Finite Difference Method

Nonstandard Finite Difference (NSFD) Method*

A numerical scheme for an initial value problem

$$\frac{d\vec{u}}{dt} = f(t, \vec{u});\ \vec{u}(0) = \vec{u}_0$$

is called a NSFD method if at least one of the following conditions is satisfied [4,5]:

(i) LHS ➔ the generalization of forward difference scheme

$$\frac{d\vec{u}_n}{dt} \approx \frac{\vec{u}_{n+1} - \vec{u}_n}{\psi(h)}$$

The nonnegative denominator function has to satisfy $\psi(h) = h + O(h^2)$, $h = \Delta t$.

(ii) The approximation of $f(t, u)$ is nonlocal.

*R. Mickens, *Nonstandard finite difference models of differential equations*, World Scientific, 1994.

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NonStandard Grünwald-Letnikov (NSGL) Approximation

$$x_{m+1} = \sum_{j=1}^{m+1} c_j^{\alpha} x_{m+1-j} + r_{m+1}^{\alpha} x_0 + s^{\alpha}\left[x_m\left(1-\frac{x_{m+1}}{\gamma}\right) - \frac{(1-c')x_{m+1}y_m}{1+\theta\xi+x_m}\right]$$

$$y_{m+1} = \sum_{j=1}^{m+1} c_j^{\alpha} y_{m+1-j} + r_{m+1}^{\alpha} y_0 + s^{\alpha}\left[\frac{\beta[(1-c')x_{m+1}+\xi]y_m}{1+\theta\xi+x_m} - \delta y_{m+1}\right]$$

$$x_{m+1} = \frac{r_{m+1}^{\alpha} x_0 + s^{\alpha} x_m + \sum_{j=1}^{m+1} c_j^{\alpha} x_{m+1-j}}{1 + s^{\alpha}\left[\frac{x_m}{\gamma} + \frac{(1-c')y_m}{1+\theta\xi+x_m}\right]}$$

$$y_{m+1} = \frac{r_{m+1}^{\alpha} y_0 + s^{\alpha}\left[\frac{\beta[(1-c')x_{m+1}+\xi]y_m}{1+\theta\xi+x_m}\right] + \sum_{j=1}^{m+1} c_j^{\alpha} y_{m+1-j}}{1+\delta s^{\alpha}}$$

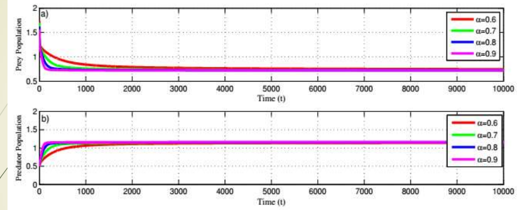

Figure 3: Numerical solution of system (2) with $\gamma = 1.5, c' = 0.22, \theta = 0.6, \xi = 0.1, \beta = 0.21, \delta = 0.08$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$: (a) Prey population $x(t)$ and (b) Predator population $y(t)$.

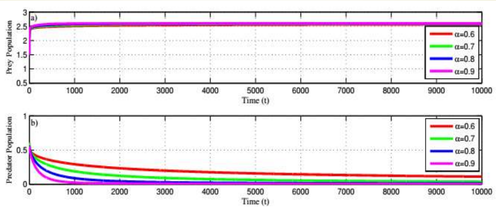

Figure 1: Numerical solution of system (2) with $\gamma = 2.6, c' = 0.6, \theta = 0.6, \xi = 0.2, \beta = 0.21, \delta = 0.08$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$: (a) Prey population $x(t)$, (b) Predator population $y(t)$.

Concluding Remarks

1. Fractional-order model for predator-prey model has been discussed: non-negativity, boundedness, existence and uniqueness of solution have been presented.
2. Model has three equilibria: extinction point, predator extinction, coexistence point. Extinction point is unstable and others are conditionally stable.

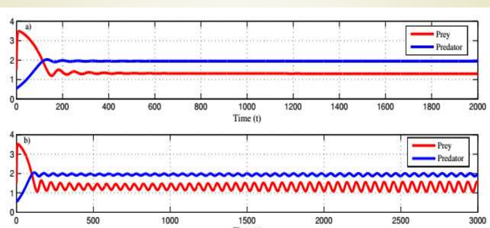

Figure 2: Numerical solution of system (2) with $\gamma = 4, c' = 0.16, \theta = 0.6, \xi = 0.2, \beta = 0.15, \delta = 0.08$ and (a) $\alpha = 0.88$, (b) $\alpha = 0.91$.

References

[1] Z. M. Odibat and N. T. Shawagfeh, Generalized Taylors formula, *Applied Mathematics and Computation* **186** (2007), 286-293, DOI: 10.1016/j.amc.2006.07.102.

[2] H. Li, L. Zhang, C. Hu, J. Yao-Lin and Z. Teng, Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge, *Journal of Applied Mathematics and Computing* **64** (2016), 435-449, DOI: 10.1007/s12190-016-1017-8.

[3] Y. Li, Y. Chen and I. Podlubny, Stability of fractional-order nonlinear dynamics: Lyapunov direct method and generalized Mittag-Leffler stability, *Computers & Mathematics with Applications* **59** (2010), 1810-1821, DOI: 10.1016/j.camwa.2009.08.019.

[4] C. V. De-León, Volterra-type lyapunov functions for fractional-order epidemic systems, *Communications in Nonlinear Science and Numerical Simulation* **24** (2015), 75-85, DOI: 10.1016/j.cnsns.2014.12.013.

[5] J. Huo, H. Zhao and L. Zhu, The effect of vaccines on backward bifurcation in a fractional order HIV model, *ISSN: 1468-1218 Nonlinear Analysis: Real World Applications* **26** (2015), 289-305, DOI: 10.1016/j.nonrwa.2015.05.014.