Spiral Optimization Algorithm

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Introduction

- Optimization problems in Engineering are often highly nonlinear, involving many different design variables under constraints.

- Such nonlinearity often results in multimodal objective function. Hence, local search algorithms such as hill-climbing or steepest descent methods of solution are not suitable to use. Thus global search algorithms should be used to obtain optimal solutions.

- Many metaheuristic algorithms have been developed to perform global search. They are constructed based on the analogy of natural phenomena such as biological evolution (genetic algorithms), birds flocking and fish schooling (particle swarm optimization).

Two characteristics of metaheuristic algorithms are : *diversification* and *intensification*

Diversification : searching for better solutions by exploring wide region coarsely.
Intensification : searching for better solutions by searching around a good solution intensively.

Diversification in the early phase during a search can find regions having a high possibility that better solution exist, while intensification in the later phase can intensively search for much better solutions in the region found in the early phase.

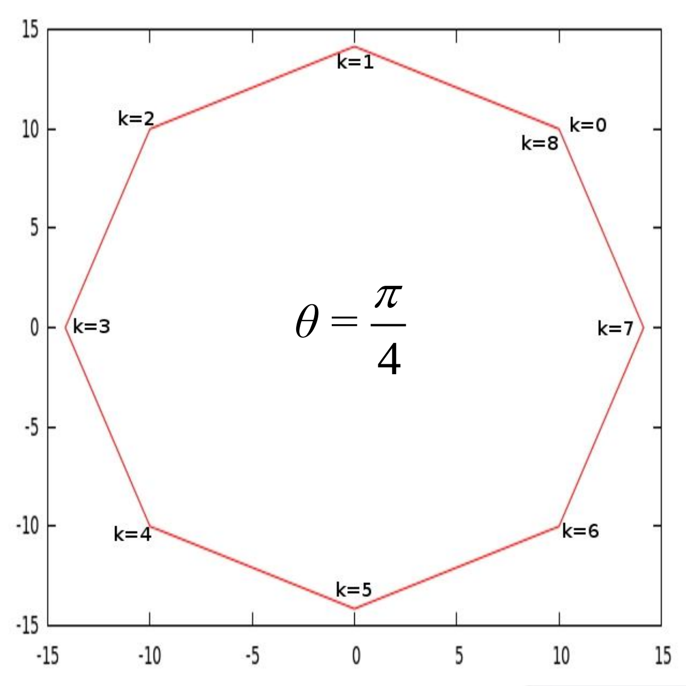
Recently a new metaheuristic search algorithm, called Spiral Dynamics Optimization, has been developed by Tamura and Yoshida (2011) of Tokyo Metropolitan University. Preliminary studies show the effectiveness of the method compared to other metaheuristics such as Particle Swarm Optimization (PSO).

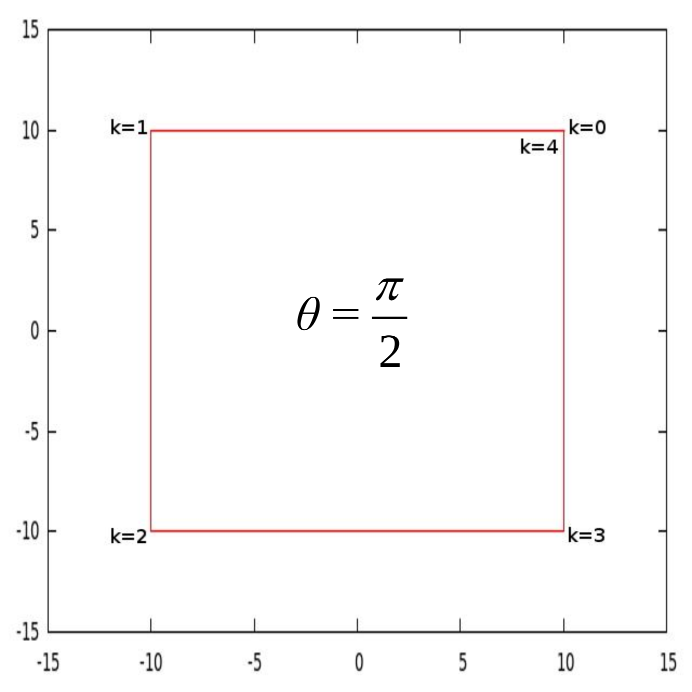
Spiral Dynamics Optimization

Rotation Model

Rotation through an angle θ

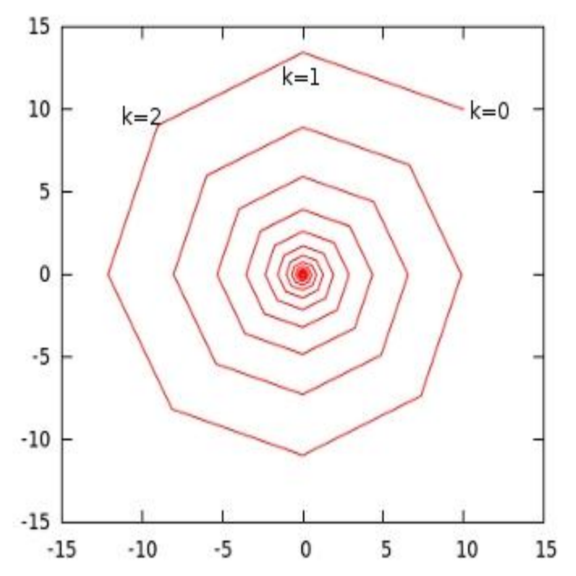
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$



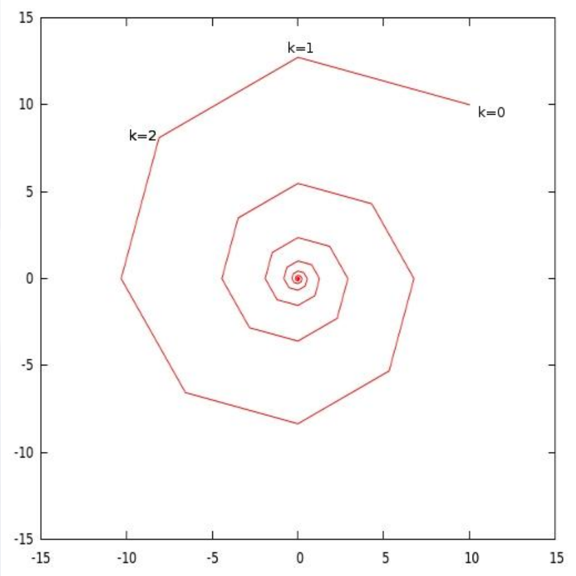


Spiral Model

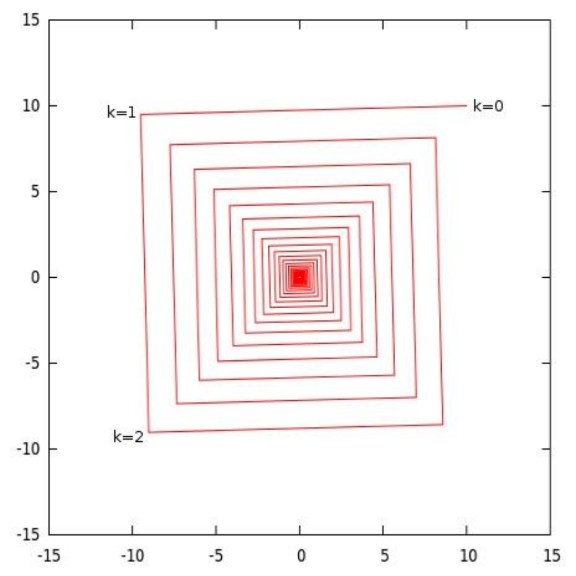
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad 0 < r < 1$$



$\theta = \frac{\pi}{4}$, $r = 0.95$



$\theta = \frac{\pi}{4}$, $r = 0.9$



$\theta = \frac{\pi}{2}$, $r = 0.95$

Spiral Model

Spiral models generate a point converging at the origin from arbitrary initial point **x** (0)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$\mathbf{x}(k+1) = S_2(r,\theta)\mathbf{x}(k) \ , \quad \mathbf{x}(0) = \mathbf{x}_0 \quad 0 \le \theta < 2\pi, \ 0 < r < 1$$

By translating the origin toward arbitrary point **x*** we have spiral model with center at **x***

$$\mathbf{x}(k+1) = S_2(r,\theta)\mathbf{x}(k) - \left(S_2(r,\theta) - I_2\right)\mathbf{x}^*$$

The trajectory will converge toward **x*** because

$$\mathbf{e}(k+1) = S_2(r,\theta)\mathbf{e}(k) \quad \text{with} \quad \mathbf{e}(k) = \mathbf{x}(k) - \mathbf{x}^*$$

Two-Dimensional Spiral Optimization

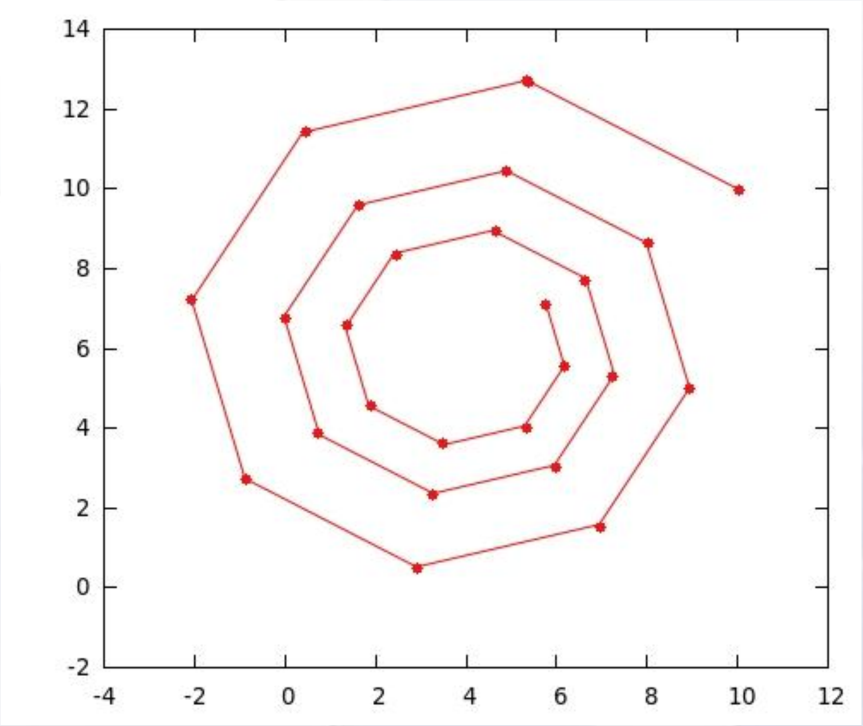
Many metaheuristics methods, such as GA, PSO, ACO use multipoint search with interaction.

The multipoint search using spiral model is formulated as

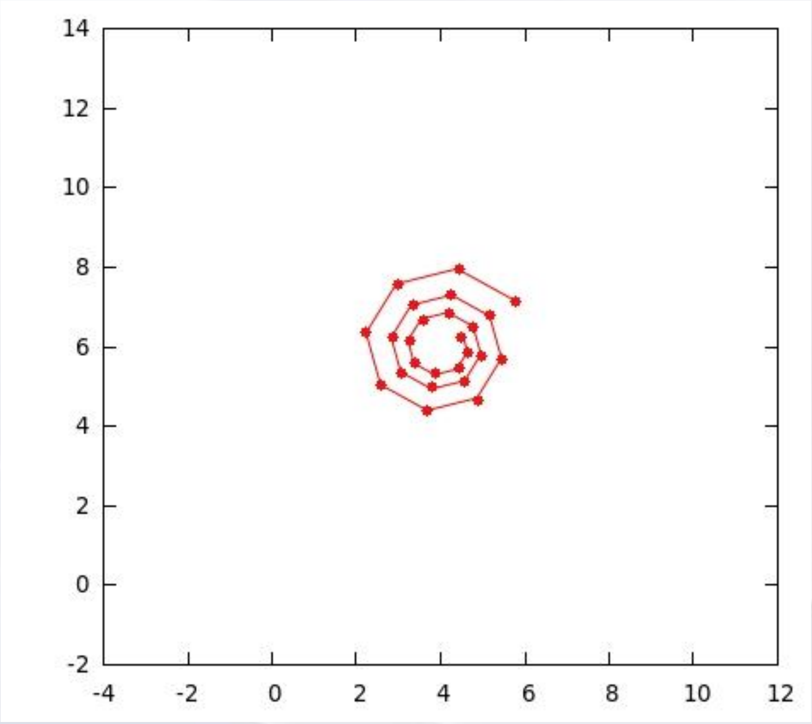
$$\mathbf{x}_i(k+1) = S_2(r,\theta)\mathbf{x}_i(k) - \left(S_2(r,\theta) - I_2\right)\mathbf{x}^* \quad i = 1, 2, \ldots, m$$

with the common center **x*** set as the best solution among all search points during a search. Thus **x*** becomes an interaction

$$\theta = \frac{\pi}{4} \quad , \quad r = 0.95$$



(a) First spiral (25 steps)



(b) Last spiral (25 steps)

$$\mathbf{x}_0 = (10,10) \quad \mathbf{x}^* = (4,6)$$

Algorithm of 2-D spiral optimization (Tamura & Yoshida (2011)) for minimization problem

Input :

$m \ (\geq 2)$ the number of search points

$\theta \ (0 \leq \theta < 2\pi), \quad r \ (0 < r < 1)$

$k_{\max}$ maximum number of iteration

Process:

1. Generate randomly initial points $\mathbf{x}_i(0) \in \mathbb{R}^2 \ \ i = 1, 2, \ldots, m$ in the feasible region.
2. Set $k = 0$

3. Find $\mathbf{x}^*$ as $\mathbf{x}^* = \mathbf{x}_{i_g}(0)$ with $i_g = \arg\min_i f\left(\mathbf{x}_i(0)\right) \; i = 1, 2, \ldots, m$

4. Update $\mathbf{x}_i$:

$$\mathbf{x}_i(k+1) = S_2(r, \theta)\mathbf{x}_i(k) - \left(S_2(r, \theta) - I_2\right)\mathbf{x}^* \quad i = 1, 2, \ldots, m$$

5. Update $\mathbf{x}^*$:

$$\mathbf{x}^* = \mathbf{x}_{i_g}(k+1), \quad i_g = \arg\min_i f\left(\mathbf{x}_i(k+1)\right), \quad i = 1, 2, \ldots, m$$

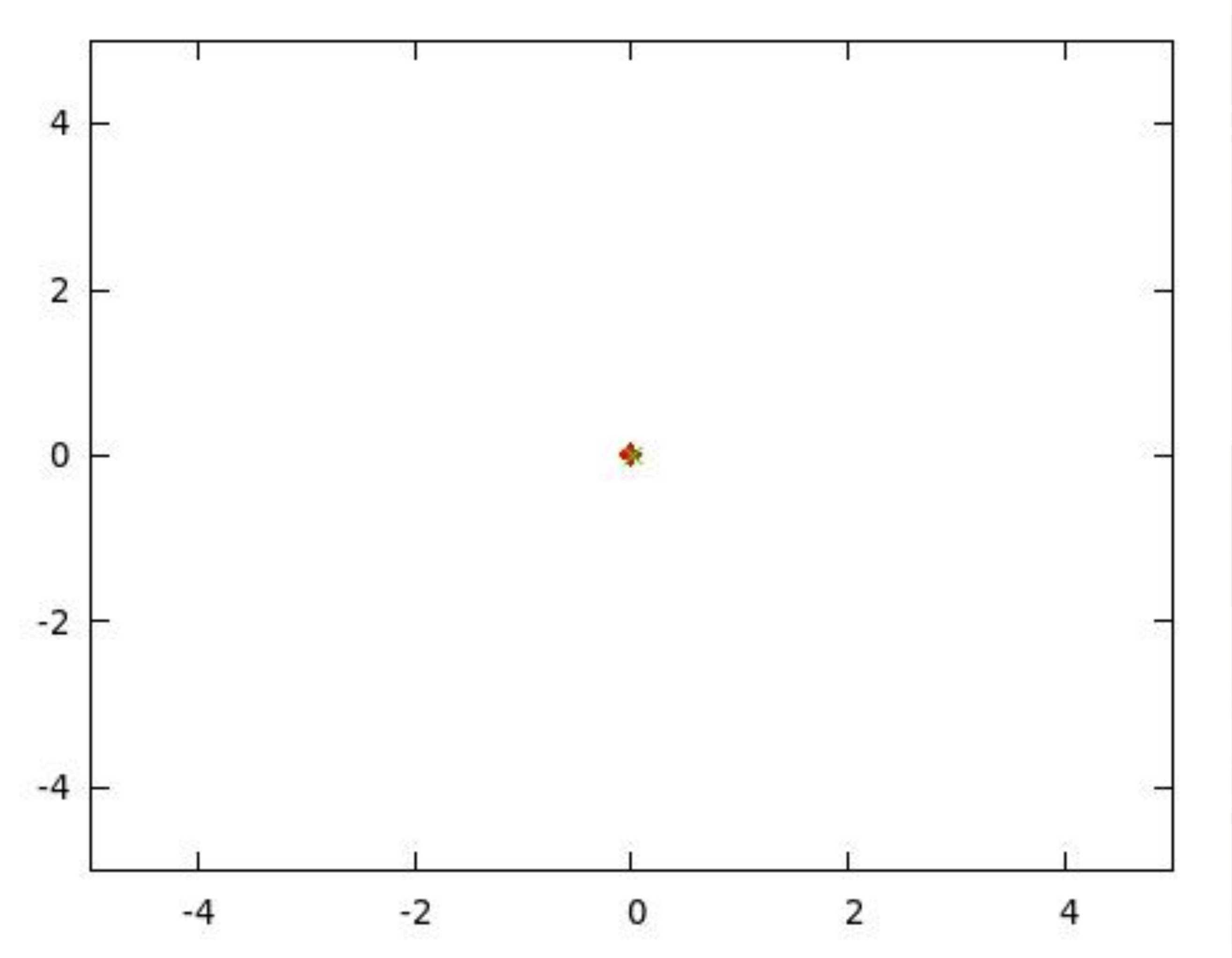
6. If $k = k_{\max}$ then terminate.

Otherwise, set $k = k + 1$ and return to step 4.

Output:

$\mathbf{x}^*$ as a minimum point of $f(\mathbf{x})$

Illustration



Function

$$f(x_1, x_2) = \frac{x_1^4 - 16x_1^2 + 5x_1}{2} + \frac{x_2^4 - 16x_2^2 + 5x_2}{2}$$

search space $- 4 £ x_1, x_2 £ 4$

Input

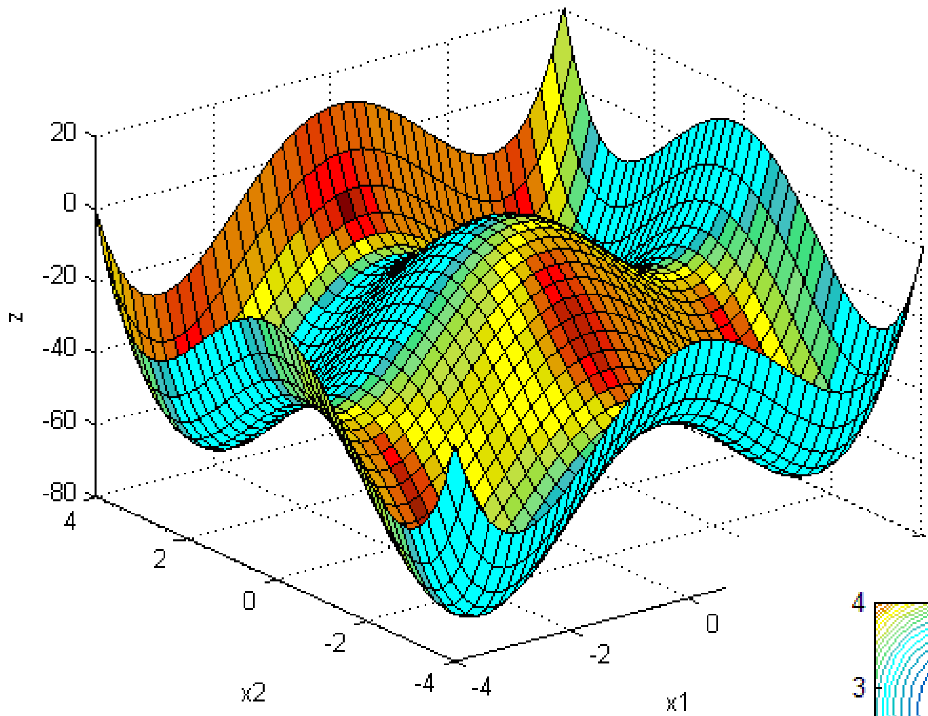
$m = 30$	$r = 0.95$
$k_{max} = 300$	$q = \frac{p}{4}$

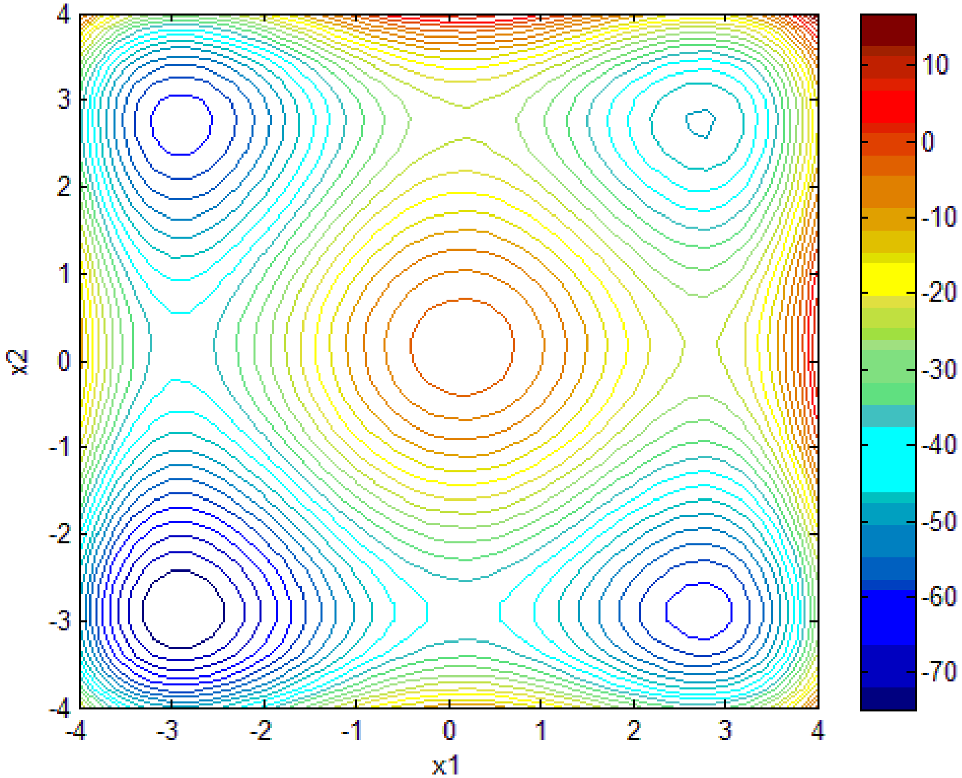
Output

$$x_1 = - 2.90353$$

$$x_2 = - 2.90353$$

$$f(x_1, x_2) = - 78.3323$$

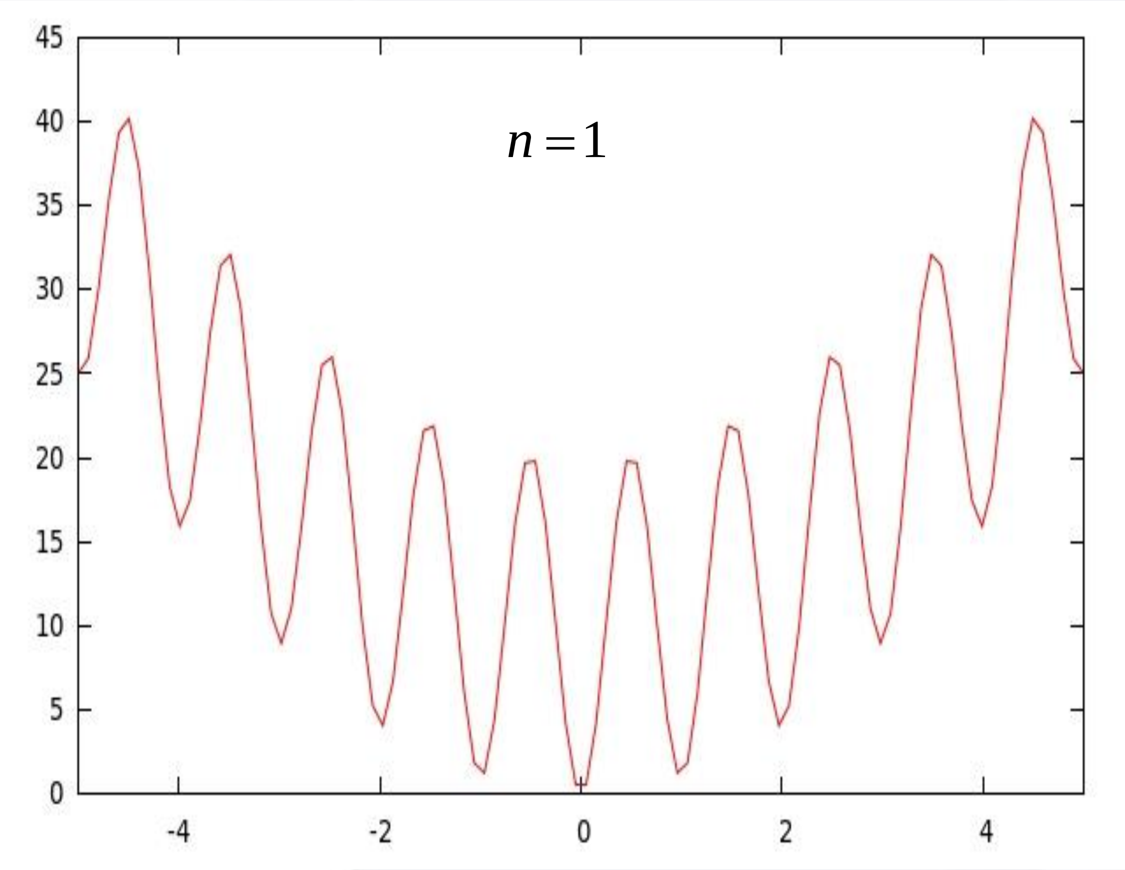

20
0
-20
-40
-60
-80
z
4
2
0
-2
-4
x2
-4
-2
0
x1

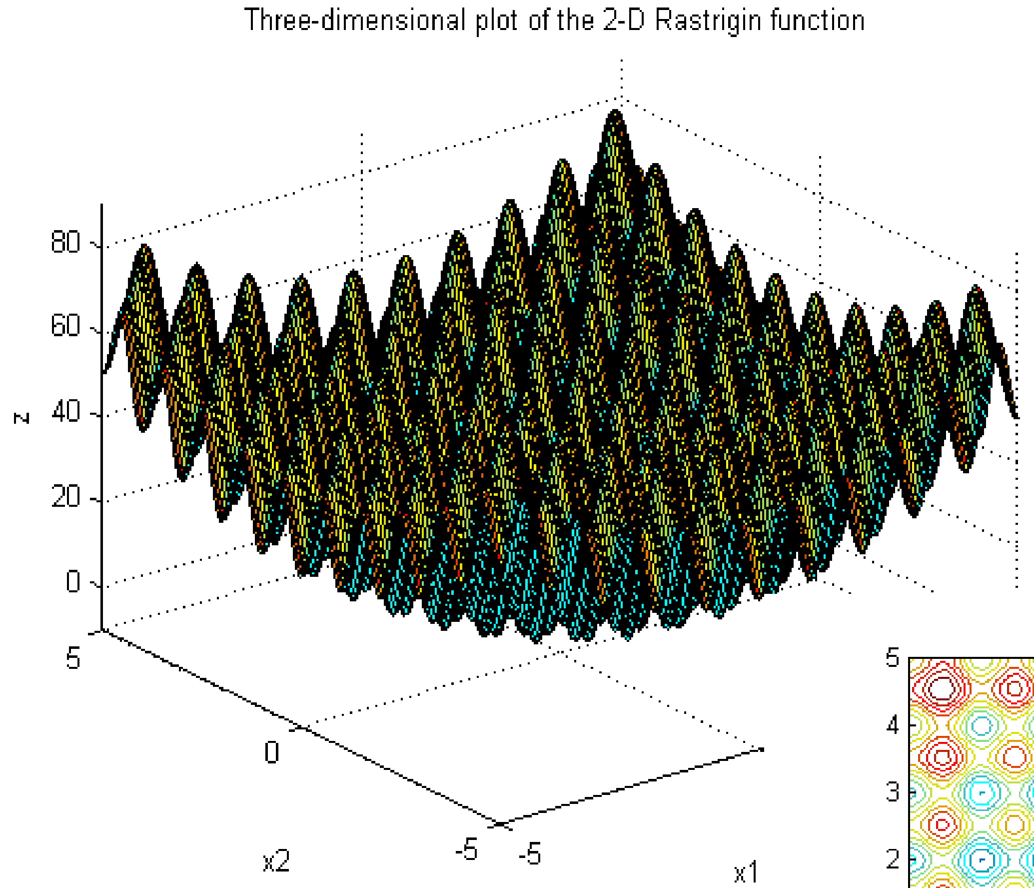

4
3
2
1
0
-1
-2
-3
-4
x2
-4
-3
-2
-1
0
1
2
3
4
x1
10
0
-10
-20
-30
-40
-50
-60
-70

Rastrigin function

$$f(\mathbf{x}) = \sum_{i=1}^{n} \left(x_i^2 - 10\cos(2\pi x_i) + 10 \right)$$

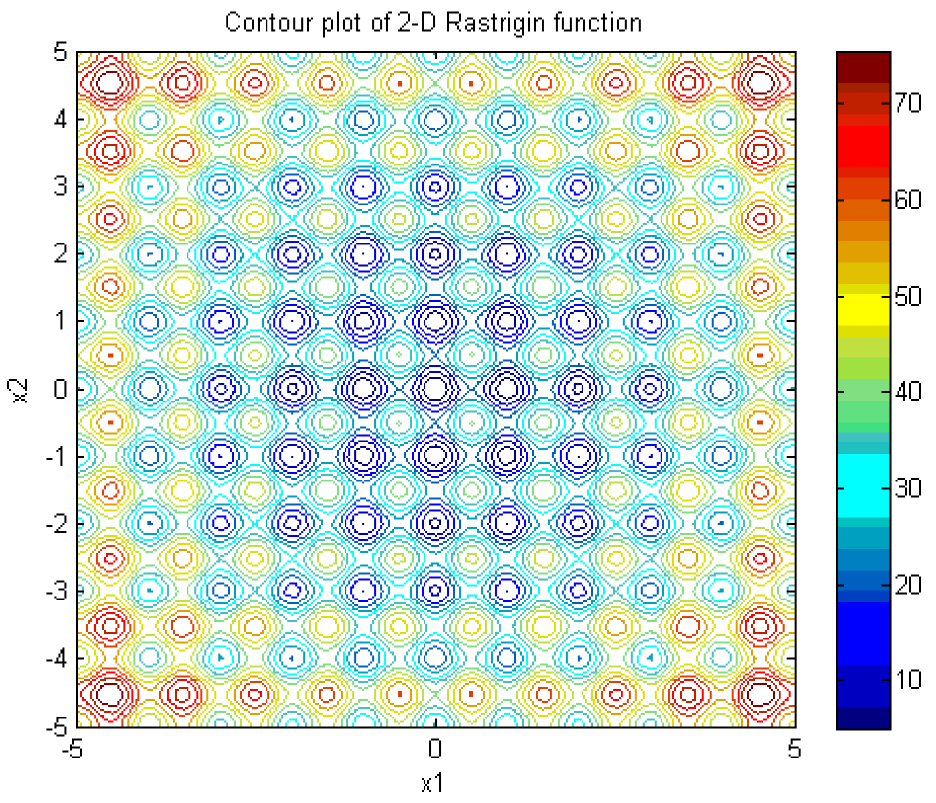
search space $\quad -5 \le x_i \le 5, \quad i = 1, 2, \ldots, n$





Three-dimensional plot of the 2-D Rastrigin function
80
60
40
20
0
z
5
0
-5
x2
-5
x1

$$n = 2$$



Contour plot of 2-D Rastrigin function
x2
x1
70
60
50
40
30
20
10

$$n = 2$$

Input

$m = 100$	$r = 0.95$
$k_{max} = 200$	$\theta = \frac{\pi}{4}$

Output

$$x_1 = 6.29823 \quad 10^{-6}$$

$$x_1 = 1.71101 \quad 10^{-6}$$

$$f\left(x_1, x_2\right) = 8.45056 \quad 10^{-9}$$

Three-Dimensional Spiral Model

$$R_{1,2}^{(3)}(\theta_{1,2}) = \begin{bmatrix} \cos\theta_{1,2} & -\sin\theta_{1,2} & 0 \\ \sin\theta_{1,2} & \cos\theta_{1,2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation in the plane $x_1 x_2$

$$R_{1,3}^{(3)}(\theta_{1,3}) = \begin{bmatrix} \cos\theta_{1,3} & 0 & -\sin\theta_{1,3} \\ 0 & 1 & 0 \\ \sin\theta_{1,3} & 0 & \cos\theta_{1,3} \end{bmatrix}$$

Rotation in the plane $x_1 x_3$

$$R_{2,3}^{(3)}(\theta_{2,3}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{2,3} & -\sin\theta_{2,3} \\ 0 & \sin\theta_{2,3} & \cos\theta_{2,3} \end{bmatrix}$$

Rotation in the plane $x_2 x_3$

Three-Dimensional Spiral Model

$$\mathbf{x}(k+1) = S_3\left(r,\theta\right)\mathbf{x}(k) \qquad 0 \le r < 1 \qquad 0 \le \theta < 2\pi$$

$$S_3\left(r,\theta\right) = rR^{(3)}(\theta_{1,2},\theta_{1,3},\theta_{2,3})$$

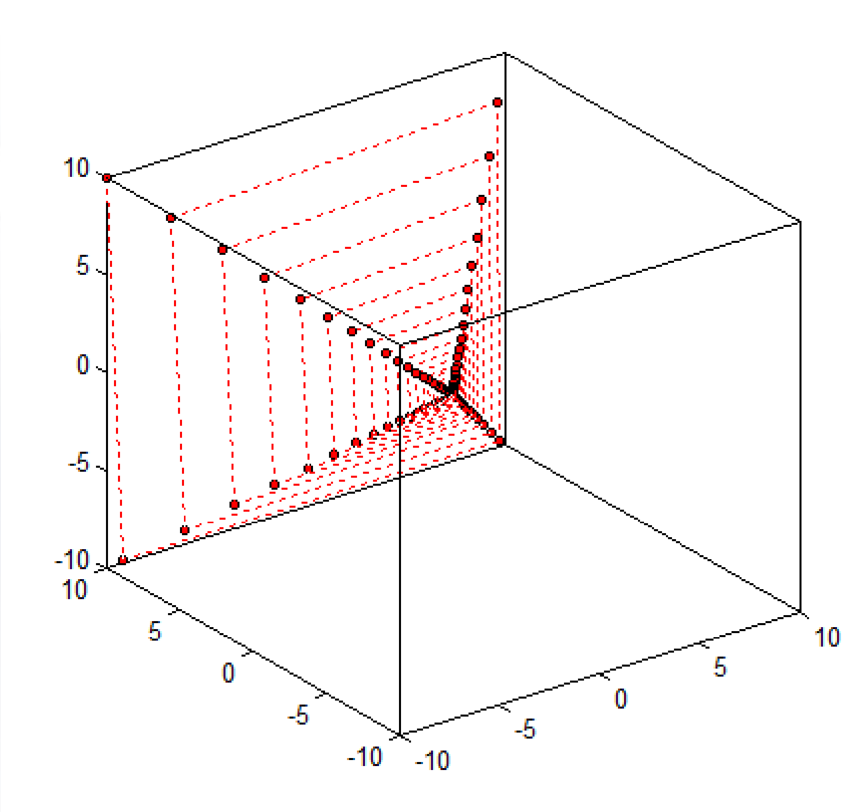
with $$R^{(3)}(\theta_{1,2},\theta_{1,3},\theta_{2,3}) := R^{(3)}_{2,3}(\theta_{2,3}) \times R^{(3)}_{1,3}(\theta_{1,3})R^{(3)}_{1,2}(\theta_{1,2})$$

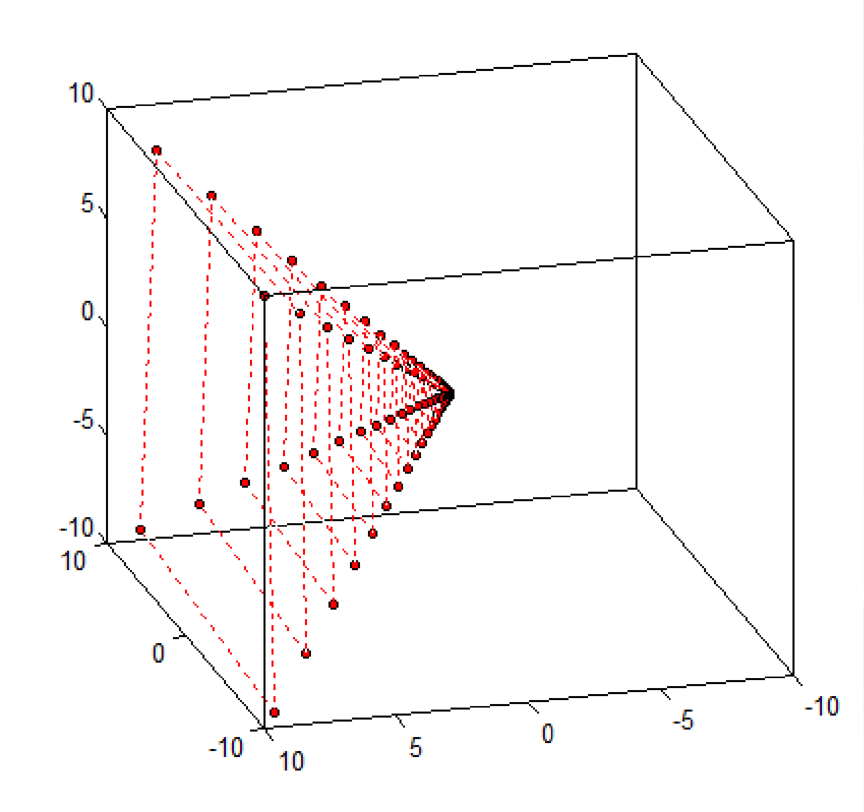
$$= \prod_{i=1}^{2}\left(\prod_{j=1}^{i} R^{n}_{3-i,3+1-j}(\theta_{3-i,3+1-j})\right)$$

with center at $\mathbf{x}_p$

$$\mathbf{x}(k+1) = S_3(r,\theta)\mathbf{x}(k) - (S_3(r,\theta) - I_3)\mathbf{x}_p$$

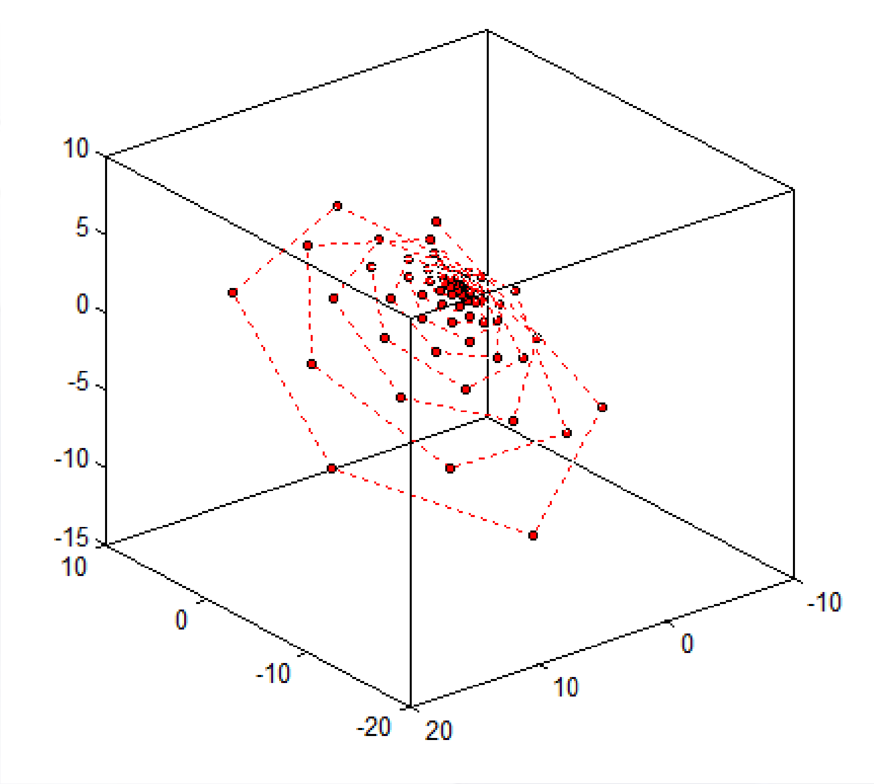
Three-Dimensional Spiral Model

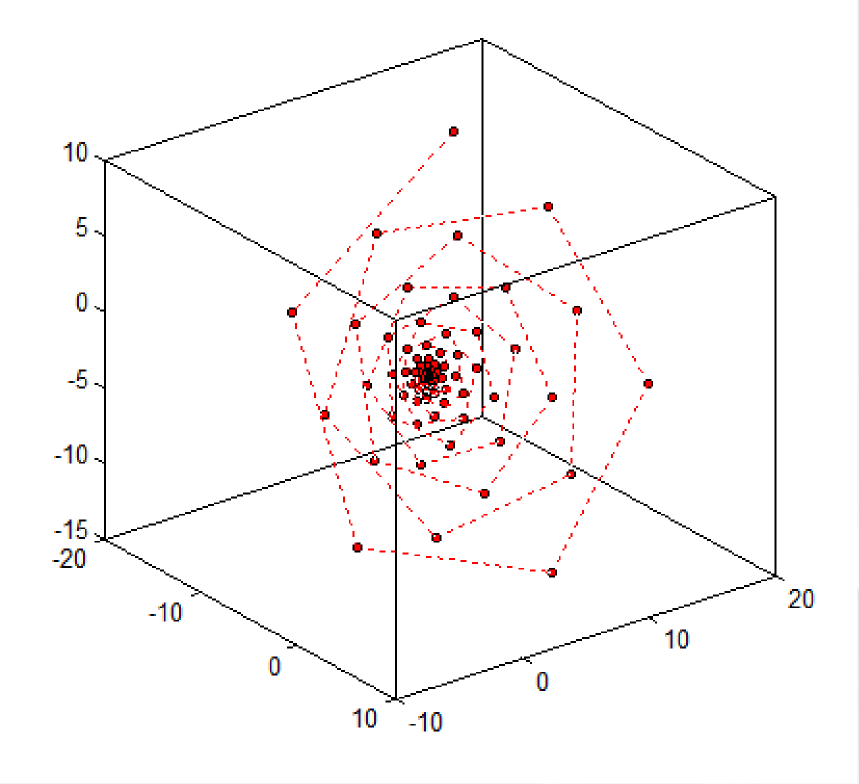




$$\theta = \frac{\pi}{2} \quad , \quad r = 0.95$$

Three-Dimensional Spiral Model





$$\theta = \frac{\pi}{4} \quad , \quad r = 0.95$$

Rotation in *n*-Dimensional Space

$$R^{(n)}(\theta) = R^{(n)}_{n-1,n}(\theta) \times R^{(n)}_{n-2,n}(\theta) \times R^{(n)}_{n-2,n-1}(\theta) \times \cdots \times R^{(n)}_{2,n}(\theta) \times \cdots \times R^{(n)}_{2,3}(\theta)$$
$$\times R^{(n)}_{1,n}(\theta) \times \cdots \times R^{(n)}_{1,3}(\theta) \times R^{(n)}_{1,2}(\theta)$$

$$R^{(n)}(\theta) = \prod_{i=1}^{n-1} \left(\prod_{j=1}^{i} R^{(n)}_{n-1,n+1-j}(\theta) \right)$$

with

$$R^{(n)}_{i,j}(\theta) = \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array} \overset{\begin{array}{ccccccccccc} & & & i & & & & j & & & \end{array}}{\begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos\theta & & & & -\sin\theta & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & \sin\theta & & & & \cos\theta & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix}}$$

n-Dimensional spiral model

$$\mathbf{x}(k+1) = S_n(r,\theta)\mathbf{x}(k) - \left(S_n(r,\theta) - I_n\right)\mathbf{x}^*$$

with $S_n(r,\theta) = r\,R^{(n)}(\theta)$

Rastrigin function

$$f(\mathbf{x}) = \sum_{i=1}^{n} \left(x_i^2 - 10\cos(2\pi x_i) + 10 \right)$$

search space $\quad -5 \le x_i \le 5, \quad i = 1, 2, \ldots, n$

$$n = 30$$

Input

$m = 100$	$r = 0.99$
$k_{max} = 1000$	$\theta = \frac{\pi}{2}$

Output

$f(\mathbf{x}) = 20.8944$

$x_1 = -0.994502$	$x_6 = -0.000157$	$x_{11} = 0.995081$	$x_{16} = -0.995011$	$x_{21} = 0.995007$	$x_{26} = -0.994909$
$x_2 = -0.994935$	$x_7 = -0.995049$	$x_{12} = 0.994926$	$x_{17} = -0.000001$	$x_{22} = 0.994472$	$x_{27} = -0.99493$
$x_3 = -0.000395$	$x_8 = 0.0002438$	$x_{13} = -0.00015$	$x_{18} = 0.994831$	$x_{23} = 0.994737$	$x_{28} = 0.0000936$
$x_4 = 0.000229$	$x_9 = -0.0003244$	$x_{14} = 0.000237$	$x_{19} = -0.0000186$	$x_{24} = -1.9901$	$x_{29} = 0.0001418$
$x_5 = 0.995131$	$x_{10} = -0.994681$	$x_{15} = 0.994778$	$x_{20} = -0.994968$	$x_{25} = 0.000165$	$x_{30} = -0.994754$

Finding Roots of Systems of Nonlinear Equations

Maximization and Root finding

Consider a system of nonlinear equation:

$$g_1(x_1, x_2, \ldots, x_n) = 0$$

$$g_2(x_1, x_2, \ldots, x_n) = 0$$

$$\vdots$$

$$g_n(x_1, x_2, \ldots, x_n) = 0$$

with $(x_1, x_2, \ldots, x_n) \in D$

$$D = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$$

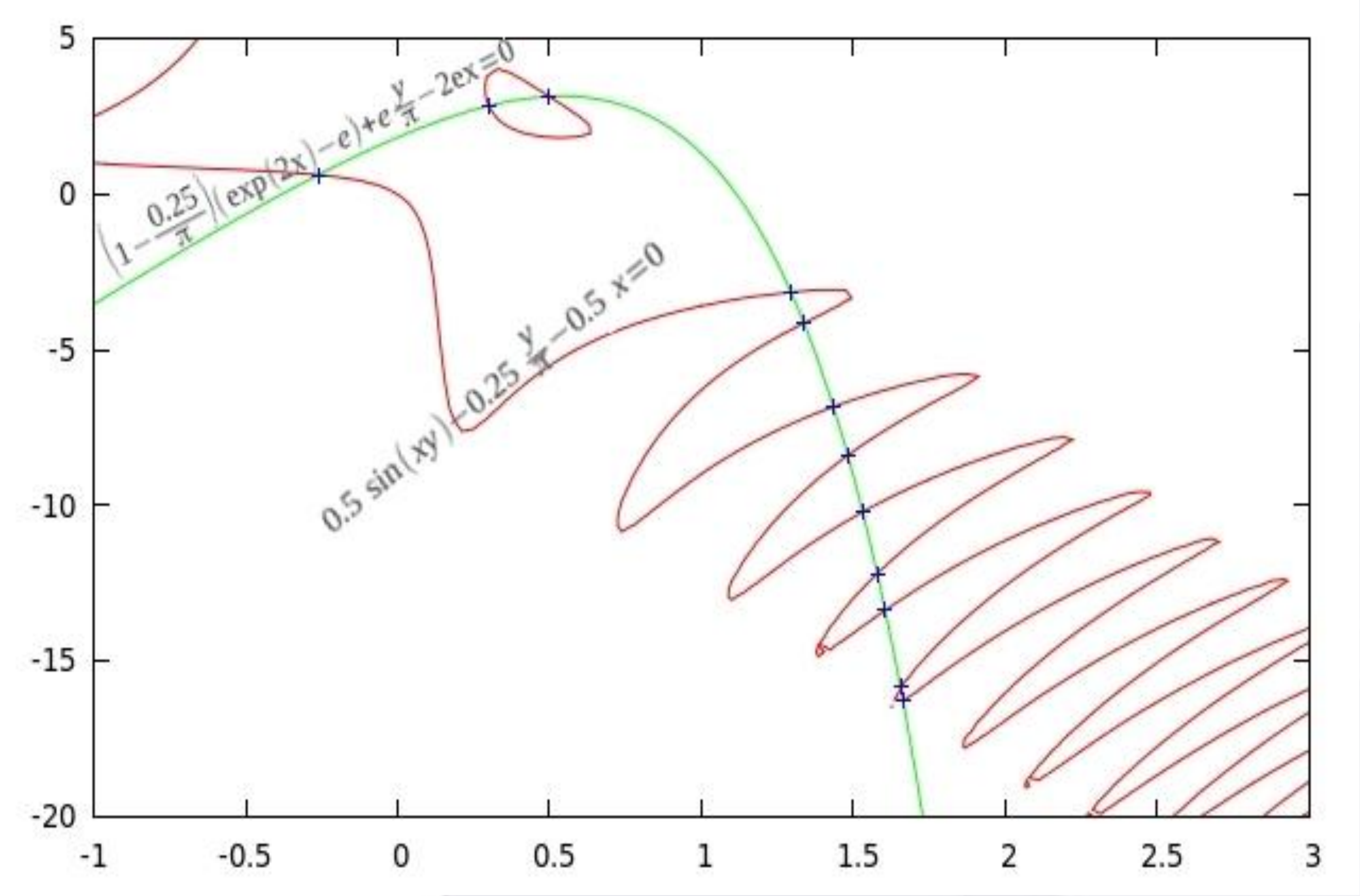
Maximization and Root finding

The above system has a solution at $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ precisely when the function $F(\mathbf{x})$ defined by :

$$F(\mathbf{x}) = \frac{1}{1 + \sum_{i=1}^{n} |g_i(\mathbf{x})|}$$

has the maximal value 1.

$$\mathbf{g}(x,y)=\begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}=\begin{bmatrix} 0.5\sin(xy)-0.25\dfrac{y}{\pi}-0.5\,x \\ \left(1-\dfrac{0.25}{\pi}\right)\left(\exp(2x)-e\right)+e\dfrac{y}{\pi}-2e\,x \end{bmatrix}=\mathbf{0}$$



Function

$$\mathbf{g} = \begin{pmatrix} x_1 + \dfrac{x_2^2 x_4 x_6}{4} + 0.75 \\ x_2 + 0.405 e^{1+x_1 x_2} - 1.405 \\ x_3 - \dfrac{x_4 x_6}{2} + 1.5 \\ x_4 - 0.605 e^{1-x_3^2} - 0.395 \\ x_5 - \dfrac{x_2 x_6}{2} + 1.5 \\ x_6 - x_1 x_5 \end{pmatrix} = \mathbf{0}$$

search space

$$-10 \le x_i \le 10$$

$$i = 1, 2, \cdots, 6$$

Input

$m = 500$	$r = 0.95$
$k_{max} = 500$	$\theta = \dfrac{\pi}{4}$

Output

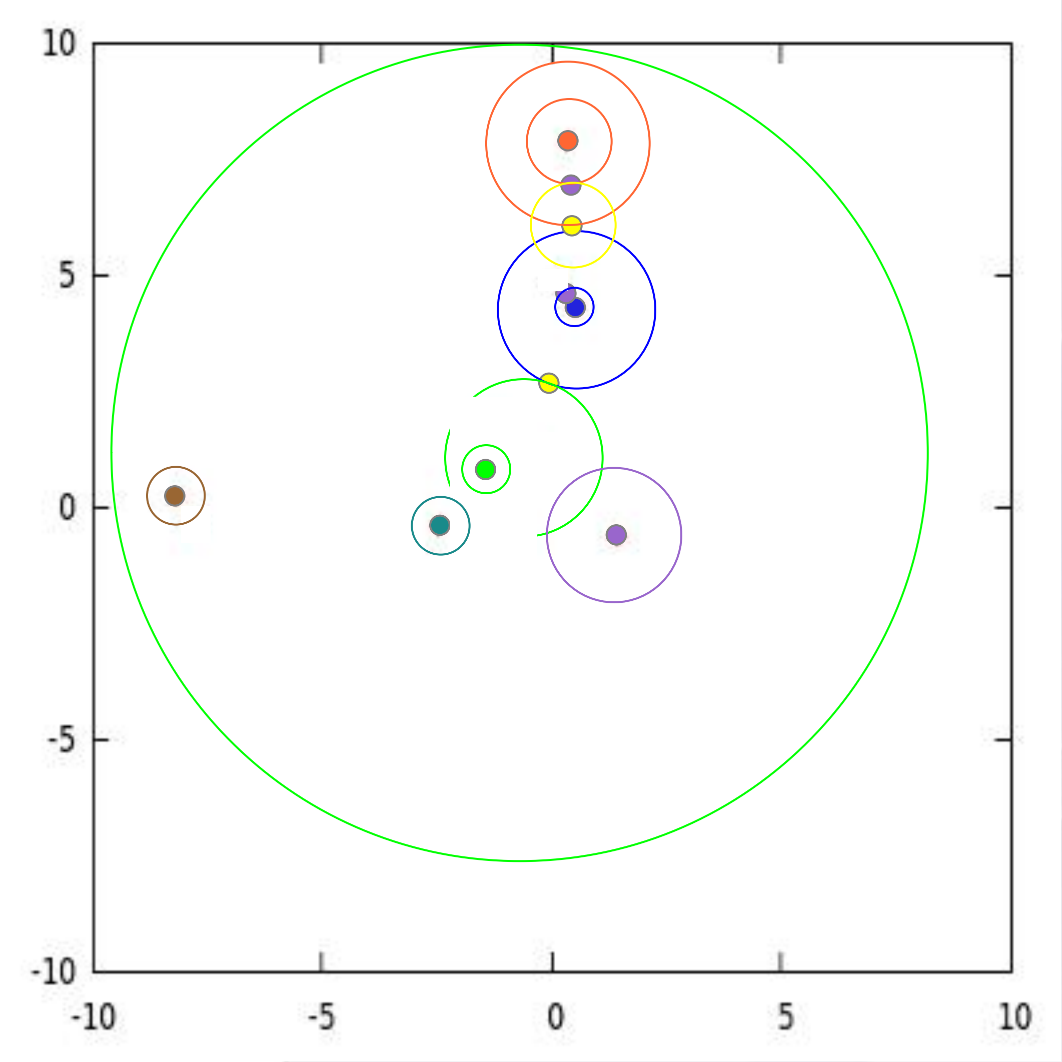
$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$F(\mathbf{x}) = 1$$

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 1.38778\,10^{-16} \\ -1.26526\,10^{-16} \\ -5.55112\,10^{-17} \\ 2.04264\,10^{-16} \\ 1.11022\,10^{-16} \\ -2.22045\,10^{-16} \end{pmatrix}$$

Clustering technique

Illustration



Algorithm

Input

$m_{cluster}\left(>1\right)$: the number of search points at the clustering phase

$\gamma\left(0<\gamma<1\right)$: 'cut-off ' parameter for function value $F\left(\mathbf{x}\right)$

$\varepsilon\left(0<\varepsilon<1\right)$: parameter for roots acceptance

$\delta\left(0<\delta<1\right)$: parameter to differentiate one candidate root from another in case they are very close each other

$k_{cluster}\left(>1\right)$: maximum iteration number at the clustering phase

$m, r, \theta, k_{\max}$: input parameters for SDOA phase

Proses

Clustering Phase

1. Generate randomly initial points $\mathbf{x}_i(0) \in \mathbb{R}^n$ $i = 1, 2, \ldots, m_{cluster}$
   in the feasible region D, where $D = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$
2. Set k=0
3. Set $\mathbf{x}'$ as $\mathbf{x}' = \mathbf{x}_{i_g}(0), \; i_g = \arg\max_i F(\mathbf{x}_i(0))$ $i = 1, 2, \ldots, m_{cluster}$
4. Store $\mathbf{x}'$ as centre of the first cluster with radius
   equal to $\frac{1}{2}\left(\min_l |b_l - a_l|\right)$ $l = 1, 2, \ldots, n$
5. For $i = 1, 2, \ldots, m_{cluster}$ do
   If $F(\mathbf{x}_i) > \gamma$ and $\mathbf{x}_i$ is not the center of already existing cluster,
   then $\mathbf{x}_i$ may have a possibility to become a cluster center,
   and then do the following functions cluster

Function Cluster (input: $\mathbf{y}$)

a. Find a cluster with center closed to $\mathbf{y}$.

b. Let C be that cluster, with center at $\mathbf{x}_C$.

c. Set $\mathbf{x}_t$ as midpoint between $\mathbf{y}$ and $\mathbf{x}_C$.

d. Compare $F(\mathbf{y}), F(\mathbf{x}_C)$ and $F(\mathbf{x}_t)$:

- If $F(\mathbf{x}_t) < F(\mathbf{y})$ and $F(\mathbf{x}_t) < F(\mathbf{x}_C)$ set a new cluster with center at $\mathbf{y}$ and radius equal the distance between points $\mathbf{y}$ and $\mathbf{x}_t$.
- Else, if $F(\mathbf{x}_t) > F(\mathbf{y})$ and $F(\mathbf{x}_t) > F(\mathbf{x}_C)$, set a new cluster with $\mathbf{y}$ as its center and radius equal to the distance between $\mathbf{y}$ and $\mathbf{x}_t$. Redo *Function Cluster* with $\mathbf{x}_t$ as its input.
- Else, if $F(\mathbf{y}) > F(\mathbf{x}_C)$, set $\mathbf{y}$ as the center of C.

e.Change the radius of C equal to the distance between $\mathbf{y}$ and $\mathbf{x}_t$.

6. Set $\mathbf{x}_p = \mathbf{x}_{i_g}$ where $i_g = \arg\max_i F\left(\mathbf{x}_i\left(k\right)\right) \ i = 1, 2, \ldots, m_{cluster}$

7. Update $\mathbf{x}_i$

$$\mathbf{x}_i\left(k+1\right) = S_n\left(r,\theta\right)\mathbf{x}_i\left(k\right) - \left(S_n\left(r,\theta\right) - I_n\right)\mathbf{x}_p \quad i = 1, 2, \ldots, m_{cluster}$$

8. Do $k_{cluster}$ times of steps 5 to 7.

Spiral Optimization Phase

9. Having done steps 1 to 8 above, we obtain a set of cluster region. Each member the set has its center and radius. To each of these cluster regions, perform SDOA to obtain a candidate of root in each cluster.

Roots Selection

10. Keep only candidate roots which satisty condition $F(\mathbf{x}) > 1 - \varepsilon$.
11. Suppose from step 10 there result n_g candidate roots. From these n_g candidate roots, select only those which satisfy $\|\mathbf{x}_i - \mathbf{x}_j\| > \delta$ for $i, j = 1, 2, \ldots, n_g$ where $\|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance between the candidate roots $\mathbf{x}_i$ and $\mathbf{x}_j$. In case where $\|\mathbf{x}_i - \mathbf{x}_j\| \le \delta$ select only $\mathbf{x}_i$ as a root if $F(\mathbf{x}_i) \ge F(\mathbf{x}_j)$, otherwise select $\mathbf{x}_j$ as a root.

Output

roots $\mathbf{x}$ of system of nonlinear equations $\mathbf{g}(\mathbf{x}) = \mathbf{0}$

Sobol Sequence

Pseudo-random may not uniformly distribute in the search feasible region of the problem

It will be helpful if it is possible to generate population of points in the search region for which the deviation from uniformity is minimal

Let $Q \subseteq [0,1]^n$

Suppose we have a set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \in [0,1]^n$

We expect that $\dfrac{\#\text{ of } \mathbf{x}_i \in Q}{\#\text{ of all points}} \approx \dfrac{vol\left(Q\right)}{vol\left([0,1]^n\right)}$

The discrepancy of the point set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ is

$$D_N = \sup_Q \left| \frac{\#\ \text{of}\ \mathbf{x}_i \in Q}{N} - vol(Q) \right|$$

When the set of rectangle is restricted to Q^* with $Q^* = \prod_{i=1}^{n} [0, y_i)$,

the corresponding discrepancy is denoted by D_n^*

The more evenly the points of a sequence are distributed,
the closer discrepancy D_N is to zero.

In this case D_N refers to the first N points of a sequence of points
$\mathbf{x}_1, \mathbf{x}_2, \ldots$. The discrepancies D_N and D_N^* satisfy $D_N^* \le D_N \le 2^n D_N^*$

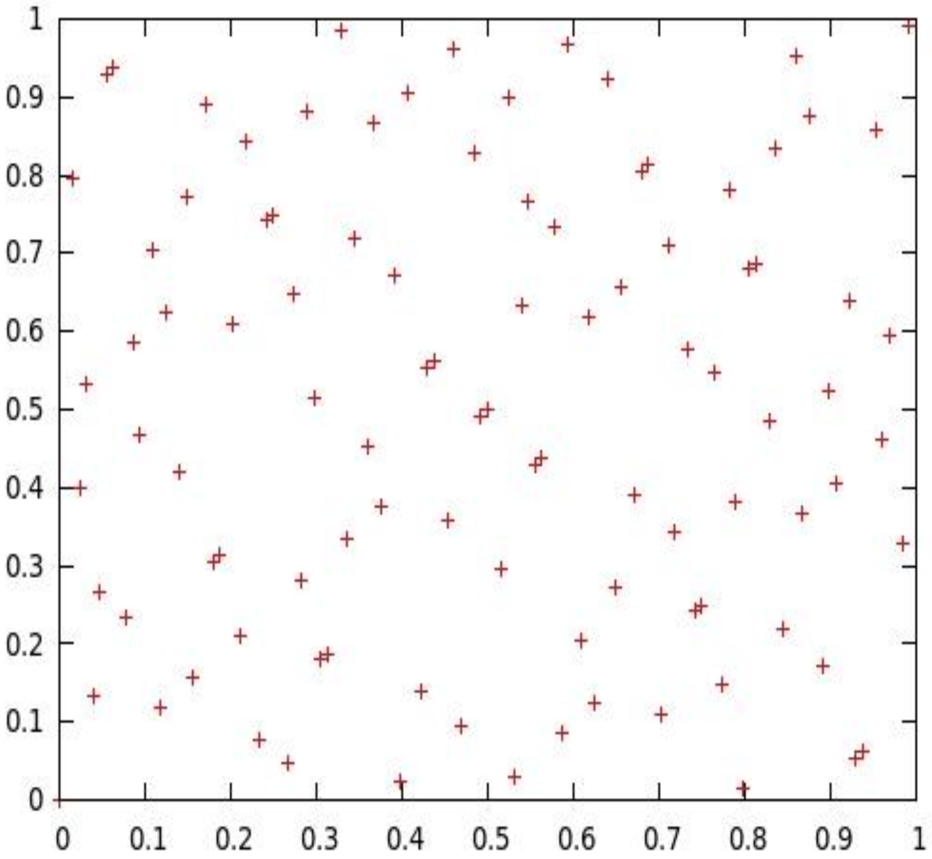
Next, a sequence of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \ldots \in \square^{n}$ is called

low-discrepancy sequence if there is a constant C_n

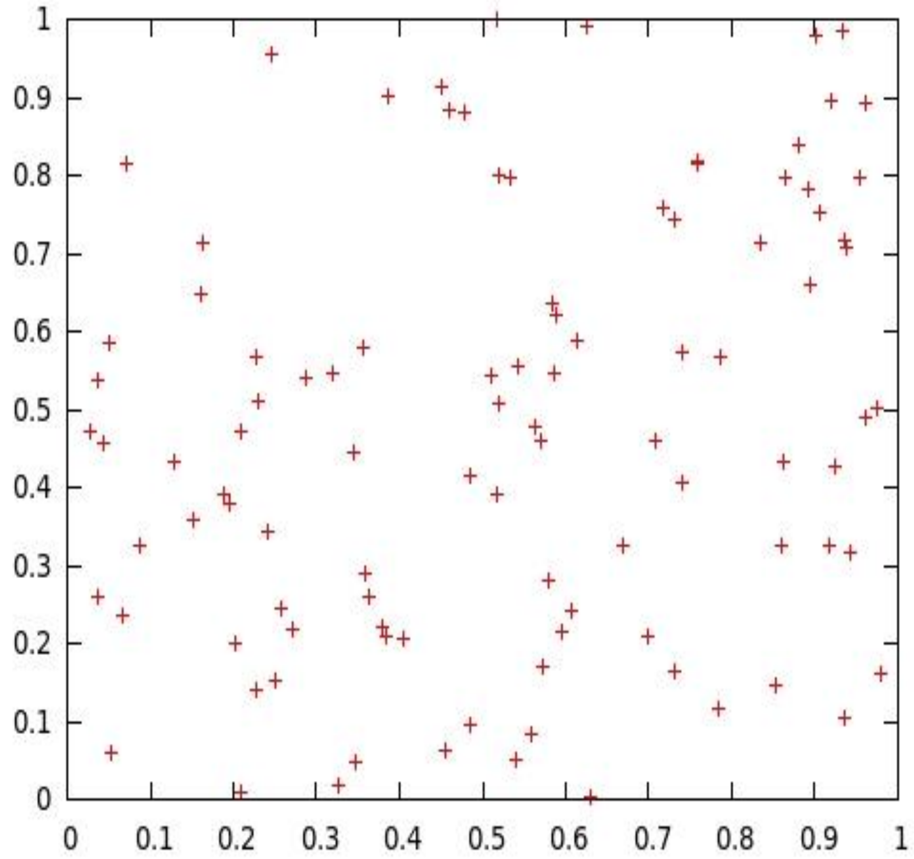
such that for all N we have $D_N \le C_n (\ln N)^n / N$

Comparison

Scatter plot of the first 100 points of :



Sobol Sequence
(quasi-random)



Pseudo-random

Numerical Experiments

Problem 1

Weierstrass Function

$$f(x) = \sum_{k=1}^{N\to\infty} \lambda^{(s-2)k} \sin\left(\lambda^k x\right)$$

where $1 < s < 2$ and $\lambda > 1$.

is known as a function which is continuous but nowhere differentiable

Use $N = 20$ with $s = 1.1$ and $\lambda = 1.5$.

$$g(x) = \sum_{k=1}^{20} 1.5^{-0.9k} \sin\left(1.5^k x\right) = 0$$

search space $0 \le x \le 5$

Results for Problem 1

Input

Clustering technique

$m_{cluster} = 200$	$\delta = 0.0001$
$k_{cluster} = 50$	$\varepsilon = 0.0000001$

$\gamma = 0.9$

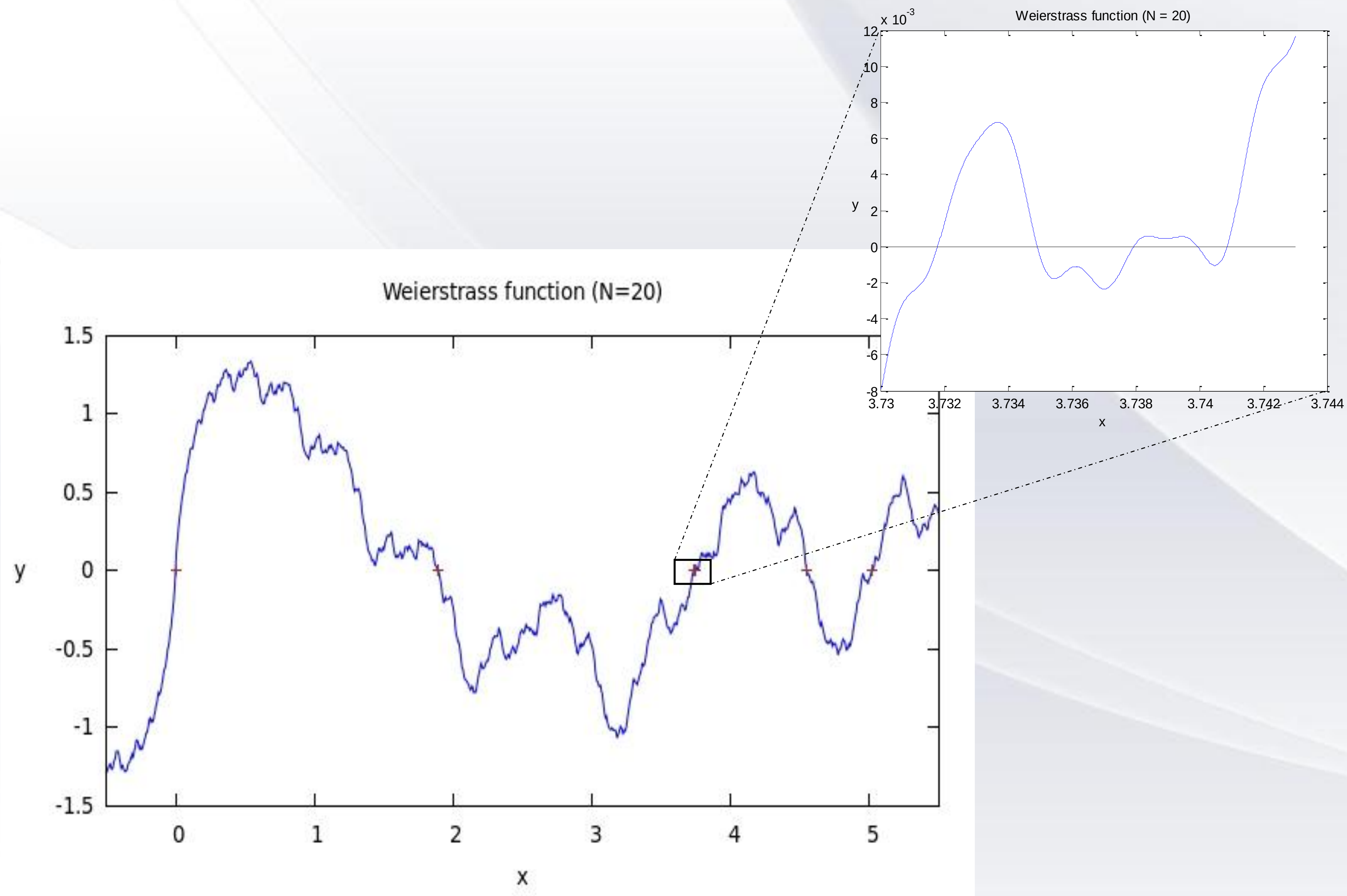
Spiral optimization

$m = 150$	$r = 0.95$
$k_{max} = 150$	$\theta = \frac{\pi}{4}$

Output

No	x	$g(x)$	No	x	$g(x)$
1	0	0	6	3.73962	9.30683e-08
2	1.88871	-9.77192e-08	7	3.74071	9.10978e-08
3	3.73173	-9.0547e-08	8	4.54986	-9.4056e-08
4	3.73499	-9.69124e-08	9	5.01996	9.56515e-08
5	3.73819	3.67608e-08			

Graph for Problem 1



Problem 2

$$\mathbf{g}(x,y)=\begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}=\begin{bmatrix} 0.5\sin(xy)-0.25\dfrac{y}{\pi}-0.5x \\ \left(1-\dfrac{0.25}{\pi}\right)\left(\exp(2x)-e\right)+e\dfrac{y}{\pi}-2ex \end{bmatrix}=\mathbf{0}$$

search space $-1 \le x \le 3,\ -17 \le y \le 4$

Input

Clustering technique

$m_{cluster} = 2000$	$\delta = 0.1$
$k_{cluster} = 10$	$\varepsilon = 0.000001$

$\gamma = 0.3$

Spiral optimization

$m = 300$	$r = 0.95$
$k_{max} = 300$	$\theta = \dfrac{\pi}{4}$

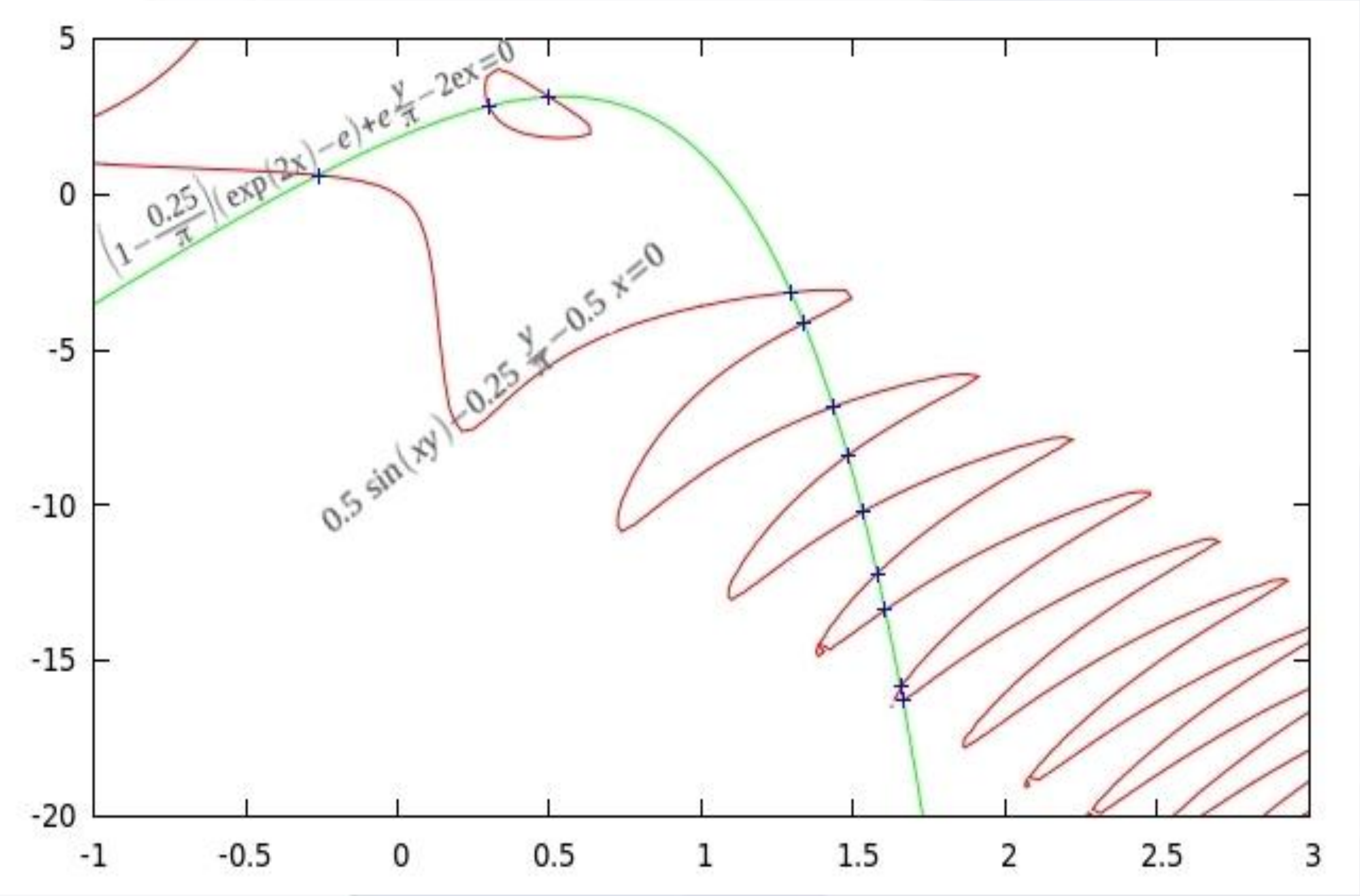
Results for Problem 2

No.	x	y	$F(x, y)$	$g_1(x, y)$	$g_2(x, y)$
1.	-0.260599	0.622531	0.999999	-1.21664e-07	-4.59048e-07
2.	0.299448	2.83693	0.999999	-1.86287e-07	4.69411e-07
3.	0.500001	3.14159	0.999999	-2.47858e-07	-4.77774e-07
4.	1.29436	-3.13722	0.999999	6.26313e-07	2.04202e-07
5.	1.33743	-4.14044	0.999999	7.58506e-07	-8.0146e-09
6.	1.43395	-6.82077	0.999999	-6.48832e-08	-5.50622e-07
7.	1.48132	-8.38361	1	1.96979e-07	1.33501e-07
8.	1.53051	-10.2022	0.999999	-1.98839e-08	5.54021e-07
9.	1.57823	-12.1767	0.999999	3.6169e-07	-4.24333e-07
10.	1.60457	-13.3629	0.999999	2.94067e-07	-2.23847e-07
11.	1.65458	-15.8192	1	-2.27255e-07	2.26023e-07
12	1.66342	-16.2828	0.999999	-9.24304e-08	-5.61499e-07

Time taken : 2.78 s

Using pseudo-random points, we have found simultaneously all roots in 6 runs from 100 runs

Graph for Problem 2



Problem 3

$$\mathbf{f}(x_1,x_2,x_3)=\begin{bmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix}=\begin{bmatrix} x_1 x_2-(x_1-2x_3)(x_2-2x_3)-165 \\ \dfrac{x_1 x_2^3}{12}-\dfrac{(x_1-2x_3)(x_2-2x_3)^3}{12}-9369 \\ \dfrac{2(x_2-x_3)^2(x_1-x_3)^2 x_3}{x_2+x_1-2x_3}-6835 \end{bmatrix}=\mathbf{0}$$

search space $-40 \le x_1, x_2, x_3 \le 40$

$$x_2 = 2x_3 - x_1 + \frac{165}{2x_3}$$

$$\mathbf{g}(x_1,x_3)=\begin{bmatrix} g_1(x_1,x_3) \\ g_2(x_1,x_3) \end{bmatrix}=\begin{bmatrix} \dfrac{x_1 x_2^3}{12}-\dfrac{(x_1-2x_3)(x_2-2x_3)^3}{12}-9369 \\ \dfrac{2(x_2-x_3)^2(x_1-x_3)^2 x_3}{x_2+x_1-2x_3}-6835 \end{bmatrix}=\mathbf{0}$$

Results for Problem 3

Input

Clustering technique

$m_{cluster} = 2000$	$\delta = 0.5$
$k_{cluster} = 10$	$\varepsilon = 0.0000001$

$\gamma = 0.001$

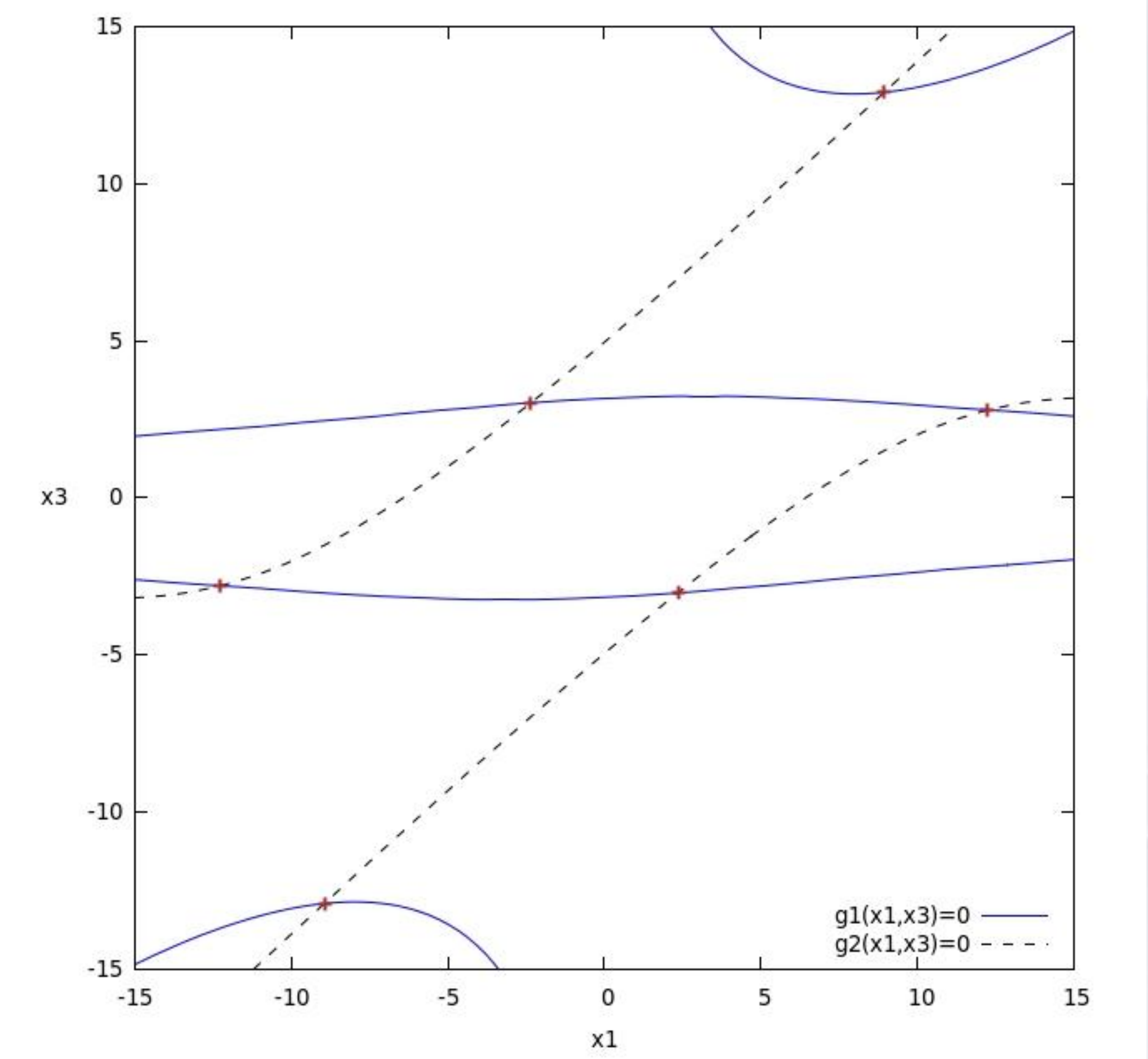
Spiral optimization

$m = 500$	$r = 0.95$
$k_{max} = 500$	$\theta = \frac{\pi}{4}$

Output

No	x_1	x_2	x_3	$f_1(x_1, x_2, x_3)$	$f_2(x_1, x_2, x_3)$	$f_3(x_1, x_2, x_3)$
1	-12.256500	-22.894900	-2.789820	-2.842170e-14	-8.305780e-04	5.900980e-05
2	-8.943090	-23.271500	-12.912800	-2.842170e-14	-5.675470e-04	-2.559920e-03
3	8.943090	23.271500	12.912800	-2.842170e-14	-5.675470e-04	-2.559920e-03
4	12.256500	22.894900	2.7898200	-2.842170e-14	-8.305780e-04	5.900980e-05
5	2.363740	-35.756400	-3.015080	-2.842170e-14	-3.012750e-03	2.403290e-04
6	-2.363740	35.756400	3.015080	-2.842170e-14	-3.012750e-03	2.403290e-04

Graph for Problem 3



Problem 4

$$\mathbf{g} = \begin{pmatrix} x_1 + \dfrac{x_2^2 x_4 x_6}{4} + 0.75 \\ x_2 + 0.405 e^{1+x_1 x_2} - 1.405 \\ x_3 - \dfrac{x_4 x_6}{2} + 1.5 \\ x_4 - 0.605 e^{1-x_3^2} - 0.395 \\ x_5 - \dfrac{x_2 x_6}{2} + 1.5 \\ x_6 - x_1 x_5 \end{pmatrix} = \mathbf{0}$$

search space $-5 \le x_i \le 5$

$i = 1, 2, \cdots, 6$

Input

Clustering technique

$m_{cluster} = 2000$	$\delta = 0.5$
$k_{cluster} = 10$	$\varepsilon = 0.0000001$

$\gamma = 0.001$

Spiral optimization

$m = 500$	$r = 0.95$
$k_{max} = 500$	$\theta = \dfrac{\pi}{4}$

Results for Problem 4

1.

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.

$$\mathbf{x} = \begin{pmatrix} -1.04320 \\ -0.550936 \\ 0.431936 \\ 1.75966 \\ -2.10487 \\ 2.19581 \end{pmatrix}$$

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} -1.737\,10^{-7} \\ -1.50569\,10^{-7} \\ -6.54736\,10^{-7} \\ 1.49601\,10^{-7} \\ -1.61856\,10^{-7} \\ 2.95125\,10^{-7} \end{pmatrix}$$

Time taken : 0.97 s

Complex Roots

real part

Complex number : $z = u + vi$

imaginary part

Complex function : $f(z) = f(u + vi)$

Example : $f(z_1, z_2) = \begin{bmatrix} f_1(z_{1,} z_2) \\ f_2(z_{1,} z_2) \end{bmatrix} = \begin{bmatrix} z_1^2 + z_2^2 + z_1 + z_2 - 8 \\ z_1 z_2 + z_1 + z_2 - 5 \end{bmatrix} = \mathbf{0}$

$$f(z_1, z_2) = \begin{bmatrix} (u_1^2 - v_1^2 + u_2^2 - v_2^2 + u_1 + u_2 - 8) + (2u_1v_1 + 2u_2v_2 + v_1 + v_2)i \\ (u_1u_2 + u_1 + u_2 - v_1v_2 - 5) + (u_1v_2 + u_2v_1 + v_1 + v_2)i \end{bmatrix} = \mathbf{0}$$

Let us consider

$$g(x_1,x_2,x_3,x_4)=\begin{bmatrix} g_1(x_1,x_2,x_3,x_4) \\ g_2(x_1,x_2,x_3,x_4) \\ g_3(x_1,x_2,x_3,x_4) \\ g_4(x_1,x_2,x_3,x_4) \end{bmatrix}=\begin{bmatrix} x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_1 + x_3 - 8 \\ 2x_1x_2 + 2x_3x_4 + x_2 + x_4 \\ x_1x_3 + x_1 + x_3 - x_2x_4 - 5 \\ x_1x_4 + x_3x_2 + x_2 + x_4 \end{bmatrix}=\mathbf{0}$$

with $D=\{-10 \le x_i \le 10, i=1,\ldots,4\}$

Input

Clustering technique	
$m_{cluster} = 300$	$\delta = 0.1$
$k_{cluster} = 100$	$\varepsilon = 0.00001$
$\gamma = 0.01$	

Spiral optimization	
$m = 100$	$r = 0.95$
$k_{max} = 300$	$\theta = \frac{\pi}{4}$

Output

Root 1

$$\mathbf{x} = \begin{pmatrix} 2 \\ 9.19285 . 10^{-7} \\ 1 \\ -2.28927 . 10^{-6} \end{pmatrix}$$

$$F(x,y) = 0.999992$$

$$z = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

real

Root 2

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1.07511 . 10^{-6} \\ 2 \\ 8.32528 . 10^{-7} \end{pmatrix}$$

$$F(x,y) = 0.999995$$

$$z = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

real

Root 3

$$\mathbf{x} = \begin{pmatrix} -3 \\ -1.41421 \\ -3 \\ 1.41421 \end{pmatrix}$$

$$F(x,y) = 0.999994$$

$$z = \begin{pmatrix} -3 - 1.41421i \\ -3 + 1.41421i \end{pmatrix}$$

complex

Root 4

$$\mathbf{x} = \begin{pmatrix} -3 \\ 1.41421 \\ -3 \\ -1.41421 \end{pmatrix}$$

$$F(x,y) = 0.999991$$

$$z = \begin{pmatrix} -3 + 1.41421i \\ -3 - 1.41421i \end{pmatrix}$$

complex

Problem : It is not always easy for a given complex function to write explicitly its real and imaginary part

Solution : Working directly with the function

In C++ we may use library *complex.h*

Problem 1

$$g(z_1,z_2)=\begin{bmatrix} g_1(z_1,z_2) \\ g_2(z_1,z_2) \end{bmatrix}=\begin{bmatrix} e^{z_1}-e^{z_2}+2 \\ z_1^3-z_2^3-1 \end{bmatrix}=\mathbf{0}$$

with

$$D=\{(z_1,z_2):-13\le u_1,u_2\le 13,-13\le v_1,v_2\le 13\}$$

Input

Clustering technique

$m_{cluster}=20000$	$\delta=0.01$
$k_{cluster}=20$	$\varepsilon=0.00001$

$\gamma=0.01$

Spiral optimization

$m=500$	$r=0.95$
$k_{max}=500$	$\theta=\frac{\pi}{4}$

Results for Problem 1

No	z_1	z_2	$g_1(z_{1,} z_2)$	$g_2(z_{1,} z_2)$
1	-0.662665+0.956196i	0.848236+0.181314i	1.96347e-06-4.71156e-07i	4.69593e-07+9.11494e-07i
2	-0.662666-0.956197i	0.848238-0.181314i	-2.88996e-06+2.18101e-06i	7.28555e-07+5.90633e-07i
3	0.680596-3.16038i	-3.10316+0.972481i	6.59986e-07+2.35274e-06i	-9.76271e-07+4.93106e-06i
4	0.680596+3.16038i	-3.10316-0.97248i	2.41556e-07+2.33136e-07i	9.89233e-07-9.29327e-06i
5	1.98471-3.05201i	1.66544+3.26502i	-1.31746e-06+1.53558e-06i	-3.36184e-06+2.4365e-06i
6	1.98471+3.05201i	1.66544-3.26502i	2.25969e-06-1.25927e-07i	2.29536e-06+6.32589e-06i

Time taken : 380.09 s

Problem 2

$$g(z_1, z_2) = \begin{bmatrix} g_1(z_1, z_2) \\ g_2(z_1, z_2) \end{bmatrix} = \begin{bmatrix} z_1^4 + 4z_2^4 - 6 \\ z_1^2 z_2 - 1.6787 \end{bmatrix} = \mathbf{0}$$

with

$$D = \{(z_1, z_2) : -2 \le u_1, u_2 \le 2, -2 \le v_1, v_2 \le 2\}$$

Input

Clustering technique

$m_{cluster} = 200$	$\delta = 0.0001$
$k_{cluster} = 50$	$\varepsilon = 0.0000001$
$\gamma = 0.9$	

Spiral optimization

$m = 150$	$r = 0.95$
$k_{max} = 150$	$\theta = \frac{\pi}{4}$

Results for Problem 2

No	z_1	z_2	$g_1(z_{1,} z_2)$	$g_2(z_{1,} z_2)$
1	1.43098-5.08152e-05i	0.819816+6.53159e-05i	-2.6359e-05-1.9781e-05i	4.4030e-05+1.4521e-05i
2	-1.4311+2.82656e-05i	0.819663+3.8692e-05i	3.7359e-05+9.5324e-06i	1.3268e-05+1.2931e-05i
3	1.39602-4.59531e-05i	0.861362+5.32735e-05i	-4.7551e-06+4.4651e-05i	-2.3060e-05-6.6922e-06i
4	-1.39615-8.3548e-05i	0.861226-9.40931e-05i	2.5552e-05-5.2203e-05i	2.5856e-05+1.7507e-05i
5	0.840215+0.840215i	1.00202e-07-1.18897i	-1.0749e-05+7.1284e-06i	2.779e-05-1.7253e-06i
6	0.840205-0.840206i	-1.62053e-06+1.18896i	-1.6171e-05+4.6572e-05i	-1.4268e-05+1.0281e-06i
7	-0.840203+0.84022i	3.68565e-06+1.18896i	-6.6519e-06-2.0486e-05i	1.2003e-05-3.8311e-05i
8	-0.840214-0.840218i	1.17022e-07-1.18897i	-9.5319e-06-1.4893e-05i	3.2477e-05+7.7607e-06i
9	-8.67857e-05-1.43107i	-0.819693-0.000114517i	-6.6203e-05-8.2757e-06i	1.1955e-06+3.0921e-05i
10	0.000159901+1.43101i	-0.819779-0.000211417i	-7.1804e-07-1.073e-05i	3.8877e-05+5.7773e-05i
11	1.8196e-05-1.39627i	-0.861096+1.63072e-05i	-5.6278e-06+3.1534e-05i	5.8816e-05+1.1963e-05i
12	6.66309e-07+1.39611i	-0.861273+2.27387e-06i	4.8289e-05-3.0496e-05i	1.6184e-05-6.0344e-06i

Problem 3

$$g(z_1,z_2)=\begin{bmatrix} g_1(z_1,z_2) \\ g_2(z_1,z_2) \end{bmatrix}=\begin{bmatrix} e^{z_1-z_2}-\sin(z_1+z_2) \\ z_1^2 z_2^2-\cos(z_1+z_2) \end{bmatrix}=\mathbf{0}$$

with

$$D=\{(z_1,z_2):-10\le u_1,u_2\le 10,-10\le v_1,v_2\le 10\}$$

Input

Clustering technique

$m_{cluster}=500$	$\delta=0.001$
$k_{cluster}=20$	$\varepsilon=0.00001$

$\delta^*=0.1$

Spiral optimization

$m=300$	$r=0.95$
$k_{max}=300$	$\theta=\frac{\pi}{4}$

Results for Problem 3

We obtained 6 real roots and 21 complex roots.

The real roots are :

No.	z_1	z_2	$g_1(z_{1,} z_2)$	$g_2(z_{1,} z_2)$
1	-0.932121+1.27131e-06i	1.06788+4.4584e-07i	-2.5591e-06-1.5896e-06i	-4.369e-07-1.643e-06i
2	0.66712-1.46425e-06i	0.690105+1.01847e-06i	-2.4946e-06-2.3318e-06i	7.2746e-07-7.4046e-07i
3	-6.43716+2.0145e-06i	0.155348-1.95019e-07i	-2.2994e-06-1.8165e-06i	6.1429e-07-3.1341e-06i
4	-6.11712+5.55163e-07i	-0.163476+2.34477e-07i	2.6471e-06-7.888e-07i	1.9554e-06-3.0481e-06i
5	0.163334-4.73192e-09i	6.12243+9.63892e-07i	-9.8743e-07-9.6166e-07i	2.1308e-06+2.5941e-07i
6	-0.155284-2.99648e-07i	6.43984+1.78116e-06i	4.0556e-07-1.4844e-06i	1.7496e-06+4.4146e-06i

Problem 4

$$\mathbf{g} = \begin{pmatrix} x_1 + \dfrac{x_2^2 x_4 x_6}{4} + 0.75 \\ x_2 + 0.405 e^{1+x_1 x_2} - 1.405 \\ x_3 - \dfrac{x_4 x_6}{2} + 1.5 \\ x_4 - 0.605 e^{1-x_3^2} - 0.395 \\ x_5 - \dfrac{x_2 x_6}{2} + 1.5 \\ x_6 - x_1 x_5 \end{pmatrix} = \mathbf{0}$$

search space $-5 \le x_i \le 5$

$i = 1, 2, \cdots, 6$

Input

Clustering technique

$m_{cluster} = 10000$	$\delta = 0.5$
$k_{cluster} = 50$	$\varepsilon = 0.001$

$\gamma = 0.1$

Spiral optimization

$m = 1000$	$r = 0.99$
$k_{max} = 1000$	$\theta = \dfrac{\pi}{4}$

Results for Problem 4

We obtained 2 real roots and 10 complex roots.

The real roots are :

No.	z	f(z)
1	$\begin{pmatrix} -0.9989317 + 0.0001109111\,i \\ 0.9991912 - 3.944306e-05\,i \\ -1.001514 - 4.49101e-05\,i \\ 0.9980045 - 5.788173e-06\,i \\ -1.000676 - 3.011193e-05\,i \\ 0.9994478 - 0.0001481381\,i \end{pmatrix}$	$\begin{pmatrix} 2.847432e-05 + 5.291092e-05\,i \\ -4.819176e-05 + 2.151123e-05\,i \\ -0.000240671 + 3.190361e-05\,i \\ -0.0001649978 + 4.847066e-05\,i \\ 4.612098e-06 + 6.360783e-05\,i \\ -0.0001588026 - 6.723182e-05\,i \end{pmatrix}$
2	$\begin{pmatrix} -1.043208 + 4.977252e-05\,i \\ -0.5510313 + 0.0001290747\,i \\ 0.4319956 - 3.65649e-05\,i \\ 1.759639 - 1.177333e-05\,i \\ -2.105187 + 0.0001195987\,i \\ 2.196163 - 5.382116e-05\,i \end{pmatrix}$	$\begin{pmatrix} 0.0001378899 - 9.680731e-05\,i \\ 0.0001069631 - 0.0001879723\,i \\ -0.0002313209 + 2.371609e-05\,i \\ 5.070828e-05 - 5.488307e-05\,i \\ -0.0001097481 - 3.696442e-05\,i \\ 1.386359e-05 + 0.0001757256\,i \end{pmatrix}$

Diophantine Equation

Modification for (Mixed) Integer Programming

Suppose among $x_1, x_2, \cdots, x_n$ variables,

$x_1, x_2, \cdots, x_i$ must be integer variables.

Before calculate $F(\mathrm{x})$ we must convert

$x_1, x_2, \cdots, x_i$ to become integer type

Problem 1

Function

$$f\left(x_1, x_2, x_3\right) = 2x_1^{\ 2} + 6x_2^{\ 3} + x_3^{\ 2} - 1825 = \mathbf{0}$$

search space $0 £ \ x_1, x_2 £ \ 50$

Input

Clustering technique

$m_{cluster} = 5000$	$\delta = 0.1$
$k_{cluster} = 10$	$\varepsilon = 0.000001$

$\gamma = 0.1$

Spiral optimization

$m = 20$	$r = 0.95$
$k_{max} = 20$	$\theta = \dfrac{\pi}{4}$

Output

No.	x_1	x_2	x_3	$f(x_1, x_2)$	$F(x_1, x_2)$
1.	15	1	37	0	1
2.	15	5	25	0	1
3.	24	2	25	0	1

Time taken : 2.47 s

Problem 2

Function

$$f\left(x, \mathrm{y}, \mathrm{c}, \mathrm{n}\right) = x^2 + 11^c - y^n = \mathbf{0}$$

search space 1£ *x*,y,c,n£ 11

Input

Clustering technique

$m_{cluster} = 3000$	$\delta = 0.1$
$k_{cluster} = 10$	$\varepsilon = 0.000001$

$\gamma = 0.3$

Spiral optimization

$m = 20$	$r = 0.95$
$k_{max} = 20$	$\theta = \frac{\pi}{4}$

Output

No.	x	y	c	n	$f(x_1, x_2)$	$F(x_1, x_2)$
1.	2	5	2	3	0	1
2.	4	3	1	3	0	1
3.	5	6	1	2	0	1

Time taken : 1.12 s

Problem 3

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 5x_1 - 6x_2 + 8x_4 - 5x_5 + 6x_6 + 10x_7 - 9x_9 + 3x_{10} + 11x_{11} - 15x_{12} + 17x_{13} + 1 \\ 7x_1 + x_2 - 4x_4 + 6x_7 - 9x_8 + 5x_9 - 12x_{10} + 3x_{11} - 7x_{12} + 8x_{13} - 26 \\ 5x_1 - 24x_2 + 32x_3 - 49x_4 + 3x_5 + 19x_6 - 21x_7 - 17x_8 + 33x_9 + 9x_{10} - 12x_{11} - x_{13} - 475 \\ 20x_1 + 27x_2 - 23x_4 - 30x_5 + 34x_6 + x_7 - 7x_9 + 11x_{10} - 28x_{11} + 4x_{12} - 36x_{13} - 103 \\ 5x_1 - 10x_3 + 2x_5 - 6x_7 - 13x_9 + 34x_{11} - 9x_{13} + 352 \\ x_2 + 22x_4 - 26x_6 - 17x_8 + 19x_{10} - 4x_{12} + 84 \\ 30x_1 + 24x_2 - 55x_3 - 15x_4 - 25x_5 + 10x_6 + 40x_7 - 10x_8 + 8x_9 - 3x_{10} - 16x_{11} + 4x_{12} - 20x_{13} - 283 \\ 5x_1 - 13x_2 + 7x_4 + x_6 - 19x_7 + 19x_8 - 2x_9 + 6x_{10} + 5x_{11} - 26x_{12} + 468 \\ x_1 + 28x_2 + 33x_3 - 100x_5 + 5x_6 + 13x_7 - x_8 - x_9 + 11x_{10} - 7x_{11} - 3x_{12} + x_{13} + 100 \\ 7x_3 - 21x_4 + 35x_5 - 42x_6 + 7x_7 + 14x_8 - 35x_9 + 28x_{10} - 7x_{11} + 14x_{12} + 56x_{13} - 329 \\ 5x_7 + 5x_8 + 10x_9 - 50x_{10} + 20x_{11} - 25x_{12} + 30x_{13} + 345 \\ 2x_1 - 4x_2 + 4x_3 - 2x_4 - 6x_5 + 8x_6 + 10x_7 + 9x_8 - 12x_9 + 20x_{10} + 6x_{11} - 30x_{12} + 16x_{13} + 78 \end{pmatrix} = \mathbf{0}$$

search space $-10 \le x_i \le 10,\ i = 1, 2, \ldots, 13$

Input

Clustering technique

$m_{cluster} = 20000$	$\delta = 0.01$
$k_{cluster} = 20$	$\varepsilon = 0.000001$

$\gamma = 0.001$

Spiral optimization

$m = 200$	$r = 0.95$
$k_{max} = 500$	$\theta = \frac{\pi}{4}$

Output

$$\mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 2 \\ -1 \\ 3 \\ 7 \\ 9 \\ -4 \\ 5 \\ 5 \\ -5 \\ 10 \\ 6 \end{pmatrix} \qquad f(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Time taken : 73.88 s

Problem 4

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 10x_1 + 3x_2 + 3x_3 + 8x_4 - 1 \\ 6x_1 - 7x_2 - 5x_4 - 2 \end{bmatrix} = \mathbf{0}$$

search space $-10 \le x_1, x_2, x_3, x_4 \le 10$

Input

Clustering technique

$m_{cluster} = 3000$	$\delta = 1$
$k_{cluster} = 30$	$\varepsilon = 0.0000001$
$\gamma = 0.1$	

Spiral optimization

$m = 100$	$r = 0.95$
$k_{max} = 100$	$\theta = \frac{\pi}{4}$

Output

No.	$\mathbf{x}$	$\mathbf{f}$	$F(\mathbf{x})$	No.	$\mathbf{x}$	$\mathbf{f}$	$F(\mathbf{x})$
1.	$\begin{pmatrix} -1 \\ -4 \\ -3 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	1	3.	$\begin{pmatrix} 2 \\ 5 \\ 2 \\ -5 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	1
2.	$\begin{pmatrix} -2 \\ -2 \\ 9 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	1	4.	$\begin{pmatrix} 3 \\ 3 \\ -10 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	1

Time taken : 0.72 s

Mixed Integer Nonlinear Programming

Problem Description

The general Mixed-Integer Non-Linear Programming (MINLP) problem can be written as

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad f(\mathbf{x})$$

$$\text{subject to} \quad g_i(\mathbf{x}) = 0 \ , \ i = 1,\dots,M$$

$$\text{and} \quad h_j(\mathbf{x}) \le 0 \ , \ j = 1,\dots,N$$

$$\mathbf{x} = \left(x_1, x_2, \dots, x_q, x_{q+1}, \dots, x_n\right)^T$$

where $x_1, x_2, \dots, x_q$ are integers for a given q.

By defining a penalty function, the above constrained problem can be transformed into an unconstrained problem. Define

$$F\left(x, \alpha_i, \beta_j\right) = f(\mathbf{x}) + \sum_{i=1}^{M} \alpha_i g_i^2(\mathbf{x}) + \sum_{j=1}^{N} \beta_j (\max(h_i(\mathbf{x}), 0))^2$$

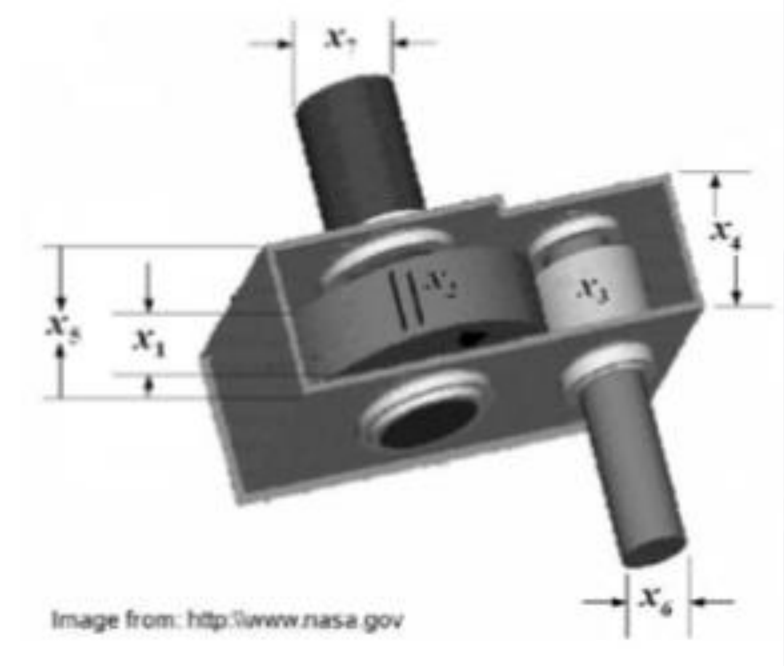
Here α_i and β_j are penalty constants. For simplicity we use the same constants, α_i and β_j for all i and j. As suggested in [4], these constants can be taken as 10^{10} to 10^{15}. In this paper we use $\alpha = \beta = 10^{15}$.

Speed Reducer Design Optimization Problem

Minimized the weight of the speed reducer
Subject to constraints on bending stress of the gear teeth, surface stress, transverse deflections of the shafts and stresses in the shaft

Design variable :

- x_1 : face width
- x_2 : module of teeth
- x_3 : number of teeth on pinion
- x_4 : length of the first shaft between bearings
- x_5 : length of the second shaft between bearings
- x_6 : diameter of the first shaft
- x_7 : diameter of the first shaft



Speed Reducer Design Optimization Problem

Minimized :

$$f(\mathbf{x}) = 0.7854 x_1 x_2^2 \left(3.3333 x_3^2 + 14.9334 x_3 - 43.0934\right)$$
$$-1.508\, x_1 (x_6^2 + x_7^2) + 7.4777 (x_6^3 + x_7^3) + 0.7854 (x_4 x_6^2 + x_5 x_7^2)$$

subject to

$$g_1(\mathbf{x}) = \frac{27}{x_1\ x_2^2\ x_3} - 1 \le 0$$

$$g_2(\mathbf{x}) = \frac{397.5}{x_1\ x_2^2\ x_3^2} - 1 \le 0$$

$$g_3(\mathbf{x}) = \frac{1.93 x_4^3}{x_2\ x_3\ x_6^4} - 1 \le 0$$

$$g_4(\mathbf{x}) = \frac{27}{x_1\ x_2^2\ x_3} - 1 \le 0$$

$$g_5(\mathbf{x}) = \frac{1.0}{110 x_6^3} \sqrt{\left(\frac{745.0 x_4}{x_2\ x_3}\right)^2 + 16.9 \times 10^6} - 1 \le 0$$

Speed Reducer Design Optimization Problem

$$g_6(\mathbf{x}) = \frac{1.0}{85x_7^3}\sqrt{\left(\frac{745.0x_5}{x_2\ x_3}\right)^2 + 157.5\times10^6} - 1 \le 0$$

$$g_7(\mathbf{x}) = \frac{x_2\ x_3}{40} - 1 \le 0$$

$$g_8(\mathbf{x}) = \frac{5x_2}{x_1} - 1 \le 0$$

$$g_9(\mathbf{x}) = \frac{x_1}{12x_2} - 1 \le 0$$

$$g_{10}(\mathbf{x}) = \frac{1.5x_6 + 1.9}{x_4} - 1 \le 0$$

$$g_{11}(\mathbf{x}) = \frac{1.1x_7 + 1.9}{x_5} - 1 \le 0$$

with

$2.6 \le x_1 \le 3.6$

$0.7 \le x_2 \le 0.8$

$17 \le x_3 \le 28$

$7.3 \le x_4 \le 8.3$

$7.8 \le x_5 \le 8.3$

$2.9 \le x_6 \le 3.9$

$5.0 \le x_7 \le 5.5$

x_3 integer

Results for Speed Reducer Problem

Penalty function :

$$F(\mathbf{x}) = f(\mathbf{x}) + M\sum_{i=1}^{11} max\left(0,\ g_i(\mathbf{x})\right) \quad \text{with} \quad M = 10^{15}$$

Input

$m = 30000$	$r = 0.99$
$k_{max} = 1000$	$\theta = \dfrac{\pi}{32}$

Output

$$\mathbf{x} = \begin{pmatrix} 3.5 \\ 0.7 \\ 17 \\ 7.3 \\ 7.8 \\ 3.35021 \\ 5.28668 \end{pmatrix}$$

$$f(\mathbf{x}) = 2996.71$$

Benchmark

$$\mathbf{x} = \begin{pmatrix} 3.5 \\ 0.7 \\ 17 \\ 7.3 \\ 7.8 \\ 3.350214 \\ 5.286683 \end{pmatrix}$$

$$f(\mathbf{x}) = 2996.348165$$

Time taken : 64.82 s

Pressure Vessel design Optimization Problem

Pressure vessel are everywhere such as champagne bottle, bottles of sparkling drink, and gas tanks. For a given volume and working pressure the basic aim of designing a cylindrical vessel is to minimize the total cost.

Pressure Vessel design Optimization Problem

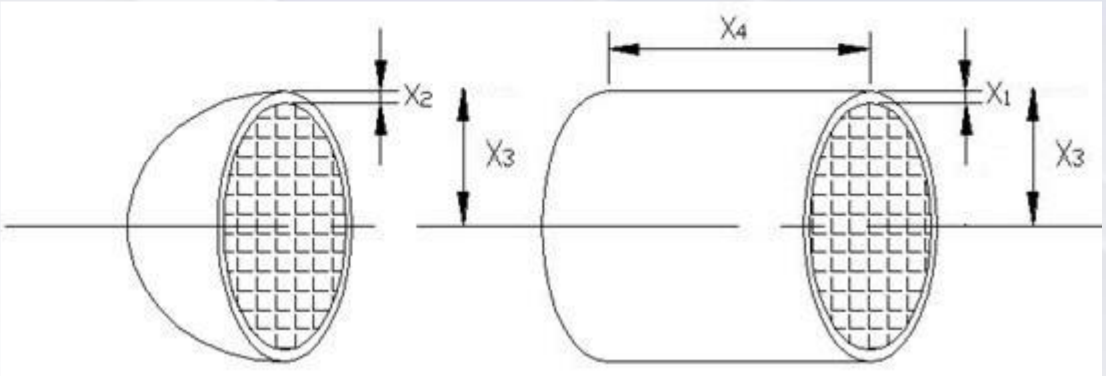
The design variables are :

x_1 : the thickness of the cylindrical shell

x_2 : the thickness of the spherical head

x_3 : the radius of the cylindrical shell

x_4 : the length of the cylindrical shell



Pressure Vessel Design Optimization Problem

$$\underset{\mathbf{x}\in\mathbb{R}^4}{\text{minimize}} \quad f(\mathbf{x}) = 0.6224\,x_1x_3x_4 + 1.7781\,x_2x_3^{\,2} + 3.1661\,x_1^{\,2}x_4 + 19.84\,x_1^{\,2}x_3$$

$$\text{subject to} \quad g_1(\mathbf{x}) = -x_1 + 0.0193\,x_3 \le 0$$

$$g_2(\mathbf{x}) = -x_2 + 0.00954\,x_3 \le 0$$

$$g_3(\mathbf{x}) = -\pi x_3^{\,2}x_4 - \frac{4}{3}\pi x_3^{\,3} + 1{,}296{,}000 \le 0$$

$$g_4(\mathbf{x}) = x_4 - 240 \le 0$$

$$\text{with} \quad 1\times 0.0625 \le x_1, x_2 \le 99\times 0.0625$$

$$x_3 \ge 10$$

$$x_4 \le 200$$

Results for Pressure Vessel Problem

Penalty function :

$$F(\mathbf{x}) = f(\mathbf{x}) + M\sum_{i=1}^{4} max\left(0,\ g_i(\mathbf{x})\right) \quad \text{with} \quad M = 10^{15}$$

Input

$m = 6000$	$r = 0.99$
$k_{max} = 750$	$\theta = \frac{\pi}{4}$

Output

$$\mathbf{x} = \begin{pmatrix} 0.8125 \\ 0.4375 \\ 42.0984 \\ 176.637 \end{pmatrix}$$

$$f(\mathbf{x}) = 6059.71$$

Time taken :
64.82 s

This result is similar with the reference

4 x 100 m Relay Race Problem

We have 4 sprinters , one for each fraction of a 4 x 100 m track and field relay. These sprinters are selected from a group of 6 eligible athletes to obtain the fastest possible team. Each of 6 eligible athletes run in each fraction, and their performance is noted in the table below.

4x100m Relay Race Problem

$$\underset{\mathbf{x}\in \mathbb{R}^6}{\text{minimize}} \quad \sum_{j=1}^{4}\sum_{i=1}^{6}\tau_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{i=1}^{6}x_{ij} = 1\,, \;\; \forall j, 1 \le j \le 4$$

$$\sum_{i=1}^{4}x_{ij} \le 1\,, \;\; \forall i, 1 \le j \le 6$$

with $x_{ij} = 1$ if the sprinter i runs the j fraction

$= 0$ otherwise

4x100m Relay Race Problem

Athlete	Fraction			
	Fraction 1	Fraction 2	Fraction 3	Fraction 4
Sprinter 1	12.27 s	11.57 s	11.54 s	12.07 s
Sprinter 2	11.34 s	11.45 s	12.45 s	12.34 s
Sprinter 3	11.29 s	11.50 s	11.45 s	11.52 s
Sprinter 4	12.54 s	12.34 s	12.32 s	11.57 s
Sprinter 5	12.20 s	11.22 s	12.07 s	12.03 s
Sprinter 6	11.54 s	11.48 s	11.56 s	12.30 s

Results for 4x100m Relay Race Problem

Input

$m = 5000$	$r = 0.95$
$k_{max} = 700$	$\theta = \frac{\pi}{4}$

Output

Fraction				Total time
Fraction 1	Fraction 2	Fraction 3	Fraction 4	
Athlete 3 (11.29 s)	Athlete 5 (11.22 s)	Athlete 1 (11.54 s)	Athlete 4 (11.57 s)	45.62 s

Time taken : 20.74 s

Multimodal Optimization

Algorithm

Input

$m_{cl}, r_{cl}, \theta_{cl}, k_{cl}$: parameters for SPO algorithm at diversification phase

$\varepsilon \left(0 < \varepsilon < 1\right)$: parameter to optimum points acceptance

$\delta \left(0 < \delta < 1\right)$: parameter to distinguish between one candidate optimum and another one in case they are very close each other

$m, r, \theta, k_{\max}$: parameters for SPO algorithm at intensification phase

Proses

Diversification Phase

1. Generate m_{cl} Sobol sequence of points as initial points

   $\mathbf{x}_i(0) \in \mathbb{R}^n$ $i = 1, 2, \ldots, m_{cl}$ in the feasible region D,

   where $D = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$

2. Set k=0

3. Set $\mathbf{x}^*$ as $\mathbf{x}^* = \mathbf{x}_{i_g}(0),\ i_g = \arg\max_i F(\mathbf{x}_i(0))$ $i = 1, 2, \ldots, m_{cl}$

4. Store $\mathbf{x}^*$ as centre of the first cluster with radius

   equal to $\frac{1}{2}\left(\min_l |b_l - a_l|\right)$ $l = 1, 2, \ldots, n$

5. For $i = 1, 2, \ldots, m_{cl}$ do

   If $\mathbf{x}_i$ is not the center of already existing cluster,

   then $\mathbf{x}_i$ may have a possibility to become a cluster center,

   and then do the following functions cluster

Function Cluster (input: $\mathbf{y}$)

a. Find a cluster with center closed to $\mathbf{y}$.

b. Let C be that cluster, with center at $\mathbf{x}_C$.

c. Set $\mathbf{x}_t$ as midpoint between $\mathbf{y}$ and $\mathbf{x}_C$.

d. Compare $F(\mathbf{y}), F(\mathbf{x}_C)$ and $F(\mathbf{x}_t)$:

- If $F(\mathbf{x}_t) < F(\mathbf{y})$ and $F(\mathbf{x}_t) < F(\mathbf{x}_C)$
  set a new cluster with center at $\mathbf{y}$ and radius equal the distance between points $\mathbf{y}$ and $\mathbf{x}_t$.
- Else, if $F(\mathbf{x}_t) > F(\mathbf{y})$ and $F(\mathbf{x}_t) > F(\mathbf{x}_C)$,
  set a new cluster with $\mathbf{y}$ as its center and radius equal to the distance between $\mathbf{y}$ and $\mathbf{x}_t$. Redo *Function Cluster* with $\mathbf{x}_t$ as its input.
- Else, if $F(\mathbf{y}) > F(\mathbf{x}_C)$, set $\mathbf{y}$ as the center of C.

e. Change the radius of C equal to the distance between $\mathbf{y}$ and $\mathbf{x}_t$.

6. Set $\mathbf{x}^* = \mathbf{x}_{i_g}$ where $i_g = \arg\max_i F(\mathbf{x}_i(k))$, $i = 1, 2, \ldots, m_{cl}$

7. Update $\mathbf{x}_i$

$$\mathbf{x}_i(k+1) = S_n(r,\theta)\mathbf{x}_i(k) - (S_n(r,\theta) - I_n)\mathbf{x}^* \ , \ i = 1, 2, \ldots, m_{cl}$$

8. Do k_{cl} times of steps 5 to 7.

Intensification Phase

9. Having done diversification phase, we obtain a number of clusters. Each cluster has its center and radius. To each cluster, perform SPO algorithm to obtain a candidate of maximum point in each cluster. Use $m, r, \theta, k_{\max}$ as input in this phase.

Final Selection

10. Keep only candidate maximum points which satisty condition

    $F(\mathbf{x}-\varepsilon) < F(\mathbf{x})$ and $F(\mathbf{x}+\varepsilon) < F(\mathbf{x})$.

11. Suppose from step 10 there result n_g candidate maximum points.

    From these n_g candidates, select only those which satisfy

    $\|\mathbf{x}_i - \mathbf{x}_j\| > \delta$ for $i, j = 1, 2, \ldots, n_g$ where $\|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance

    between the candidate roots $\mathbf{x}_i$ and $\mathbf{x}_j$. In case where

    $\|\mathbf{x}_i - \mathbf{x}_j\| \le \delta$ select only $\mathbf{x}_i$ as a maximum point if

    $F(\mathbf{x}_i) \ge F(\mathbf{x}_j)$, otherwise select $\mathbf{x}_j$ as a maximum point.

Output

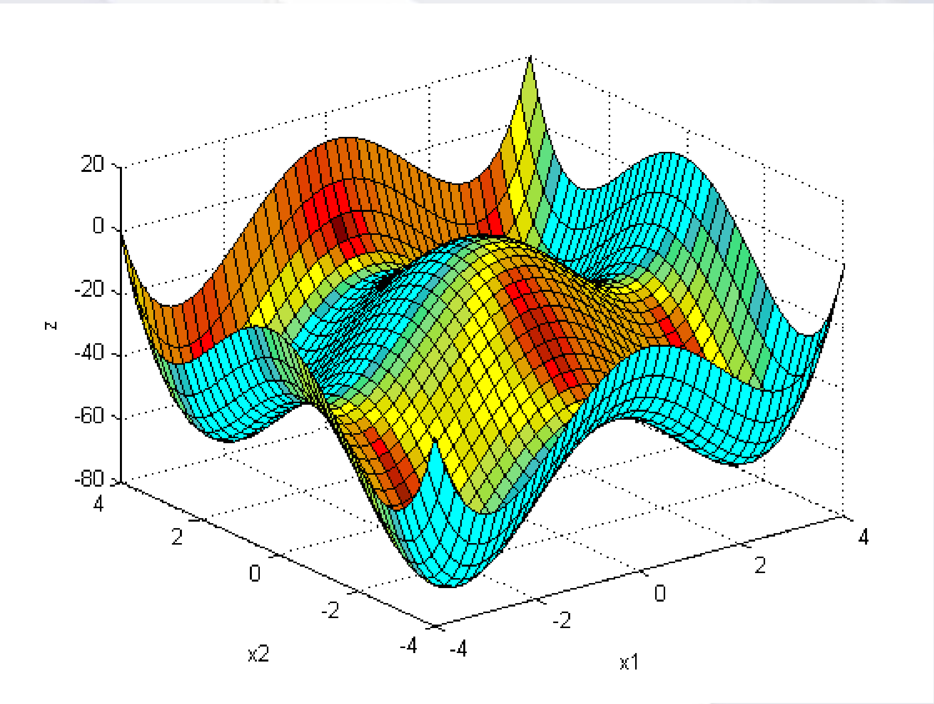
all candidates from step 9 that meet final selection become maximum points

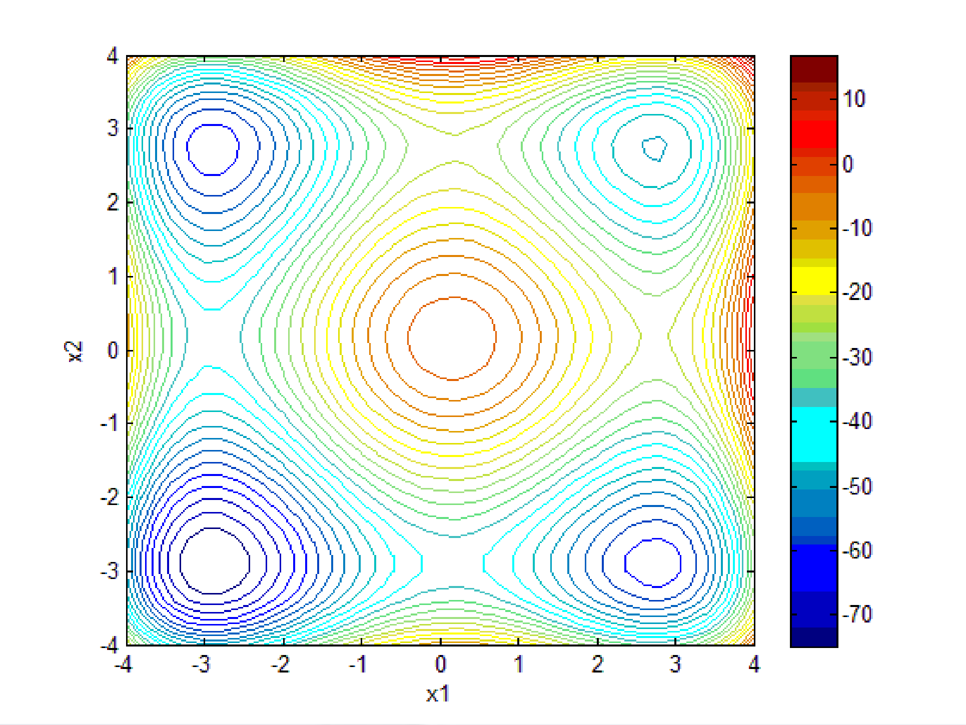
Problem 1

Function

$$f(x_1, x_2) = \frac{x_1^4 - 16x_1^2 + 5x_1}{2} + \frac{x_2^4 - 16x_2^2 + 5x_2}{2}$$

search space $-4 \le x_1, x_2 \le 4$





Input

Diversification Phase	
$m_{cl} = 300$	$r_{cl} = 0.95$
$k_{cl} = 10$	$\theta_{cl} = \frac{\pi}{4}$

Intensification Phase	
$m = 200$	$r = 0.95$
$k_{max} = 200$	$\theta = \frac{\pi}{4}$

Acceptance Parameter $\delta = 0.1$ $\varepsilon = 0.0000001$

Output

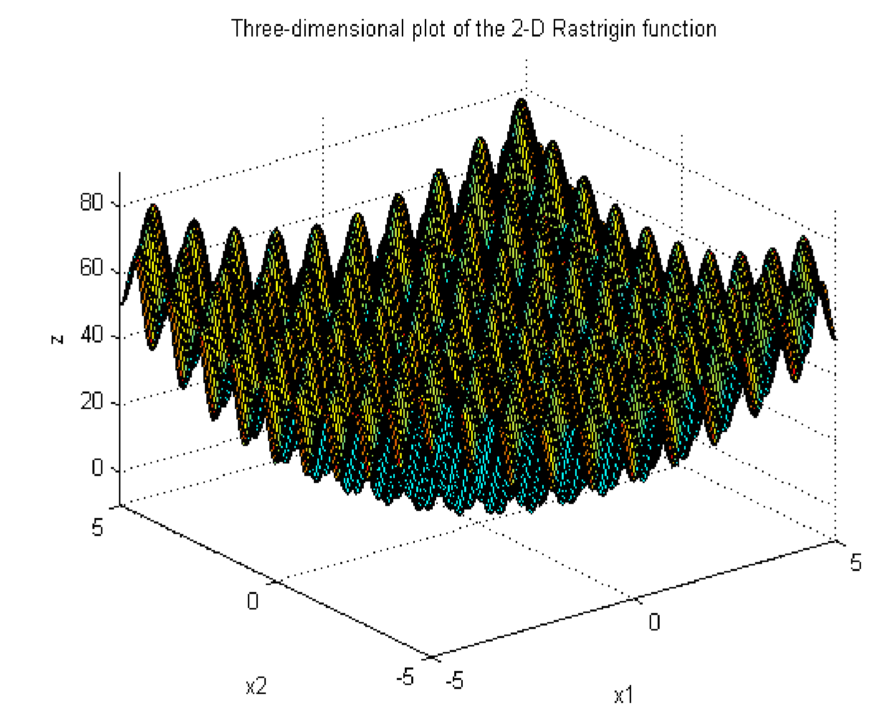
No	X	y	g(x,y)
Minimum			
1	-2.90353	-2.90353	-78.3323
2	-2.90353	2.7468	-64.1956
3	2.7468	-2.90353	-64.1956
Maximum			
1	0.156731	0.156731	0.391225

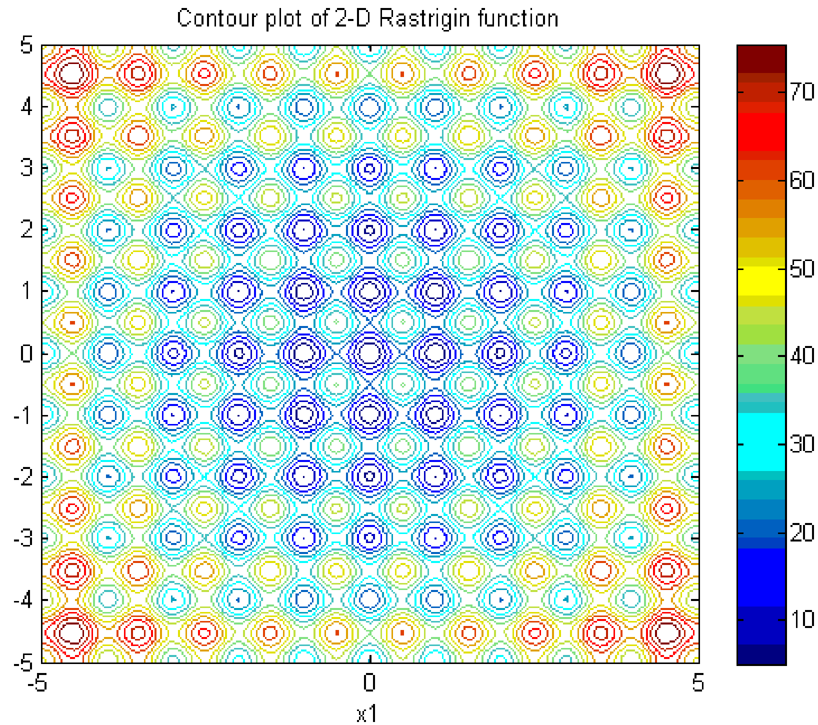
Problem 2

Function

$$f(\mathbf{x}) = \sum_{i=1}^{2} \left(x_i^2 - 10\cos(2\pi x_i) + 10 \right)$$

search space $-5 \le x_i \le 5, \quad i = 1, 2, \ldots, n$





Input

Diversification Phase

$m_{cl} = 500$	$r_{cl} = 0.95$
$k_{cl} = 10$	$\theta_{cl} = \frac{\pi}{4}$

Intensification Phase

$m = 200$	$r = 0.95$
$k_{max} = 200$	$\theta = \frac{\pi}{4}$

Acceptance Parameter

$\delta = 0.1$ $\varepsilon = 0.000001$

Output

No	x	y	F(x)
Minimum			
1	-0.994959	-0.994959	1.98992
2	-0.994959	1.6366e-09	0.994959
3	-0.994959	0.994959	1.98992
4	1.06667e-09	-0.994959	0.994959
5	0	0	0
6	-5.10989e-09	0.994959	0.99495
7	0.994959	-0.994959	1.98992
8	0.994959	1.20281e-09	0.994959
9	0.994959	0.994959	1.98992
Maximum			
1	-0.502545	-0.502546	40.5025
2	-0.502537	0.502538	40.5025
3	0.502546	-0.502545	40.5025
4	0.502548	0.502544	40.5025

Finding Max / Min of multi-modal functions

$$g = g(\mathbf{x}) = g\left((x_1, x_2, \ldots, x_n)^T\right)$$

- Its stationary points are the roots of

$$\nabla g(\mathbf{x}) = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}^T = \mathbf{0}^T$$

- Let $\mathbf{x}_0$ be a stationary point of g
- The Hessian matrix of g at $\mathbf{x}_0$: $H = H(\mathbf{x}_0)$

  with $h_{ij} = \frac{\partial^2 g}{\partial x_i \, \partial x_j}(\mathbf{x}_0)$

- Then : (i) $\mathbf{x}_0$ is a local minimum of g if $H(\mathbf{x}_0)$ is definit positif

  (ii) $\mathbf{x}_0$ is a local maximum of g if $H(\mathbf{x}_0)$ is definit negatif

  (iii) $\mathbf{x}_0$ is a saddle point of g if $H(\mathbf{x}_0)$ is indefinit

Illustration :

 $z = g\left(x, y\right) = \frac{1}{2}\left(x^4 - 16x^2 + 5x\right) + \frac{1}{2}\left(y^4 - 16y^2 + 5y\right)$

 Its stationary points are the roots of :

$$\frac{\partial g}{\partial x} = \frac{1}{2}\left(4x^3 - 32x + 5\right) = 0$$

$$\frac{\partial g}{\partial y} = \frac{1}{2}\left(4y^3 - 32y + 5\right) = 0$$

 Its Hessian matrix is: $H\left(x, y\right) = \begin{pmatrix} 6x^2 - 16 & 0 \\ 0 & 6y^2 - 16 \end{pmatrix}$

$$g(x,y)=\frac{1}{2}\left(x^4-16x^2+5x\right)+\frac{1}{2}\left(y^4-16y^2+5y\right)$$

Search space

$$D=\left\{(x,y):-4\le x\le 4,\,-4\le y\le 4\right\}$$

$$F(x,y)=\frac{1}{1+\left|\frac{1}{2}\left(4x^3-32x+5\right)\right|+\left|\frac{1}{2}\left(4y^3-32y+5\right)\right|}$$

Input

Clustering technique

$m_{cluster}=200$	$\delta=0.1$
$k_{cluster}=100$	$\varepsilon=0.00001$

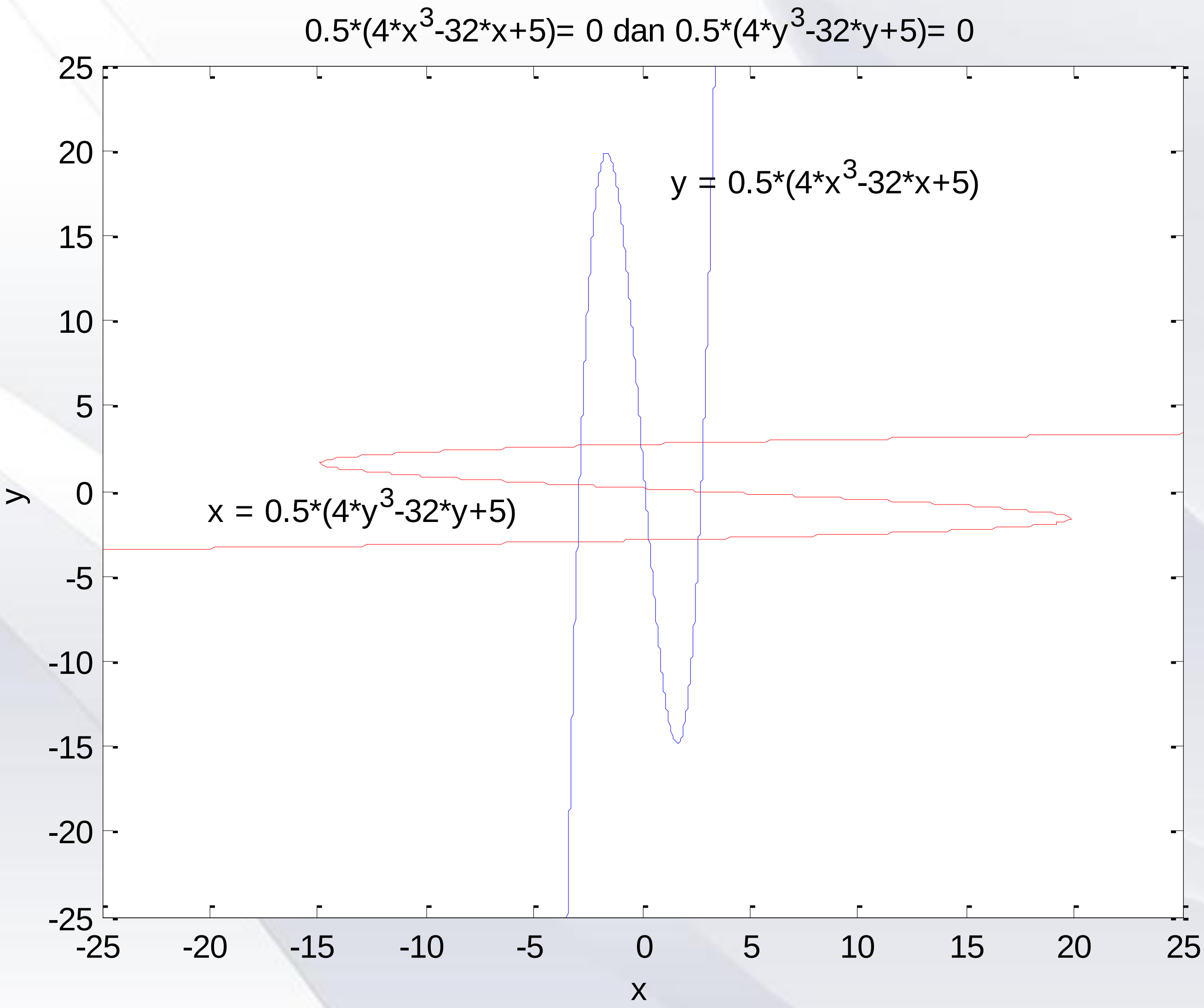
Spiral optimization

$m=100$	$r=0.95$
$k_{max}=300$	$\theta=\frac{\pi}{4}$

Illustration

No	x	y	$F(x,y)$	$g(x,y)$
1	-2.90353	-2.90353	0.999996	-78.3323
2	-2.90353	0.156732	0.999993	-38.9706
3	-2.90353	2.7468	0.999994	-64.1956
4	0.156731	-2.90353	0.999993	-38.9706
5	0.156731	0.156731	0.999995	0.391225
6	0.156731	2.7468	0.999996	-24.8338
7	2.7468	-2.90353	0.999998	-64.1956
8	2.7468	0.156731	0.999994	-24.8338
9	2.7468	2.7468	0.999992	-50.0589

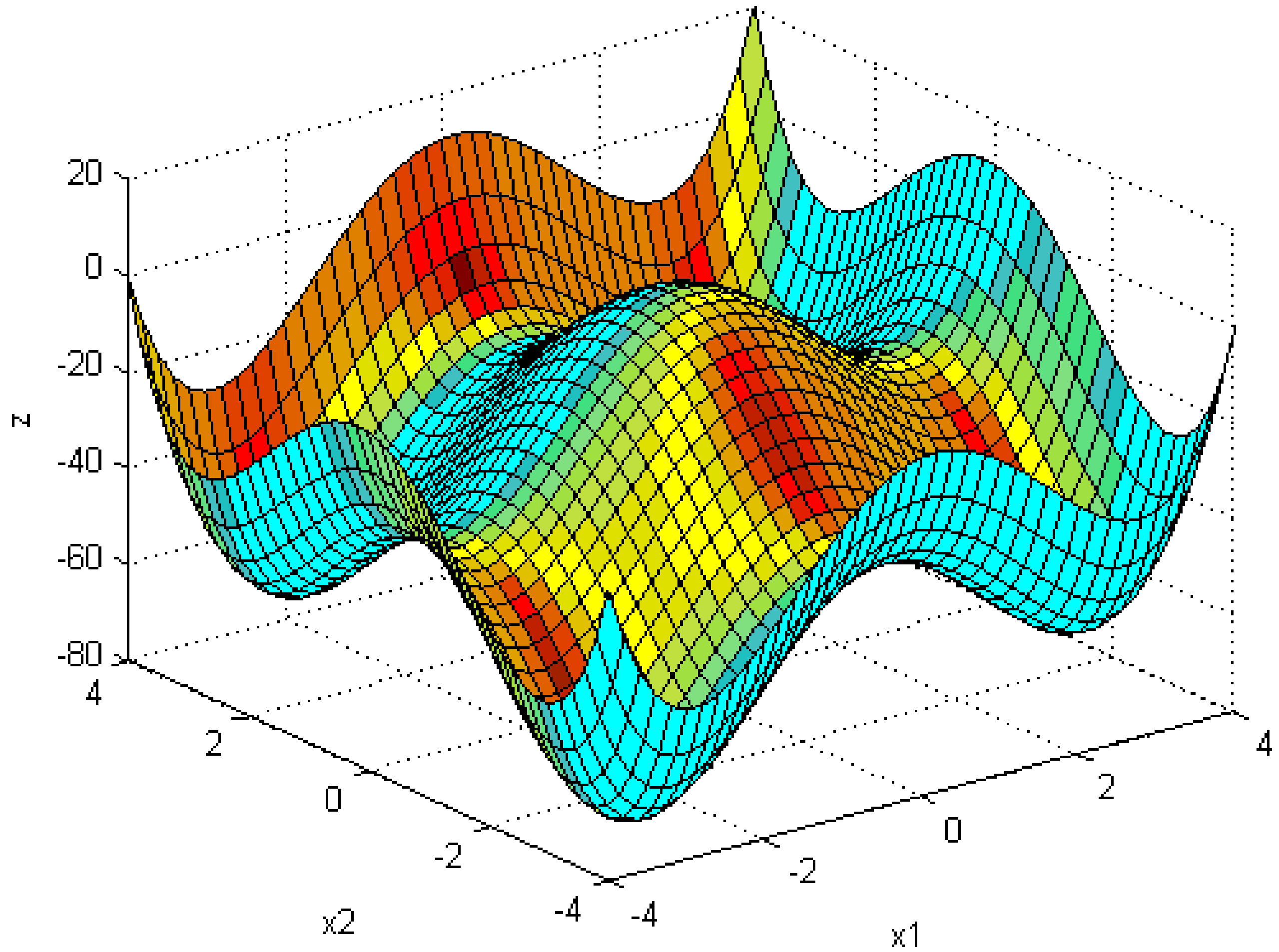
Stationary points



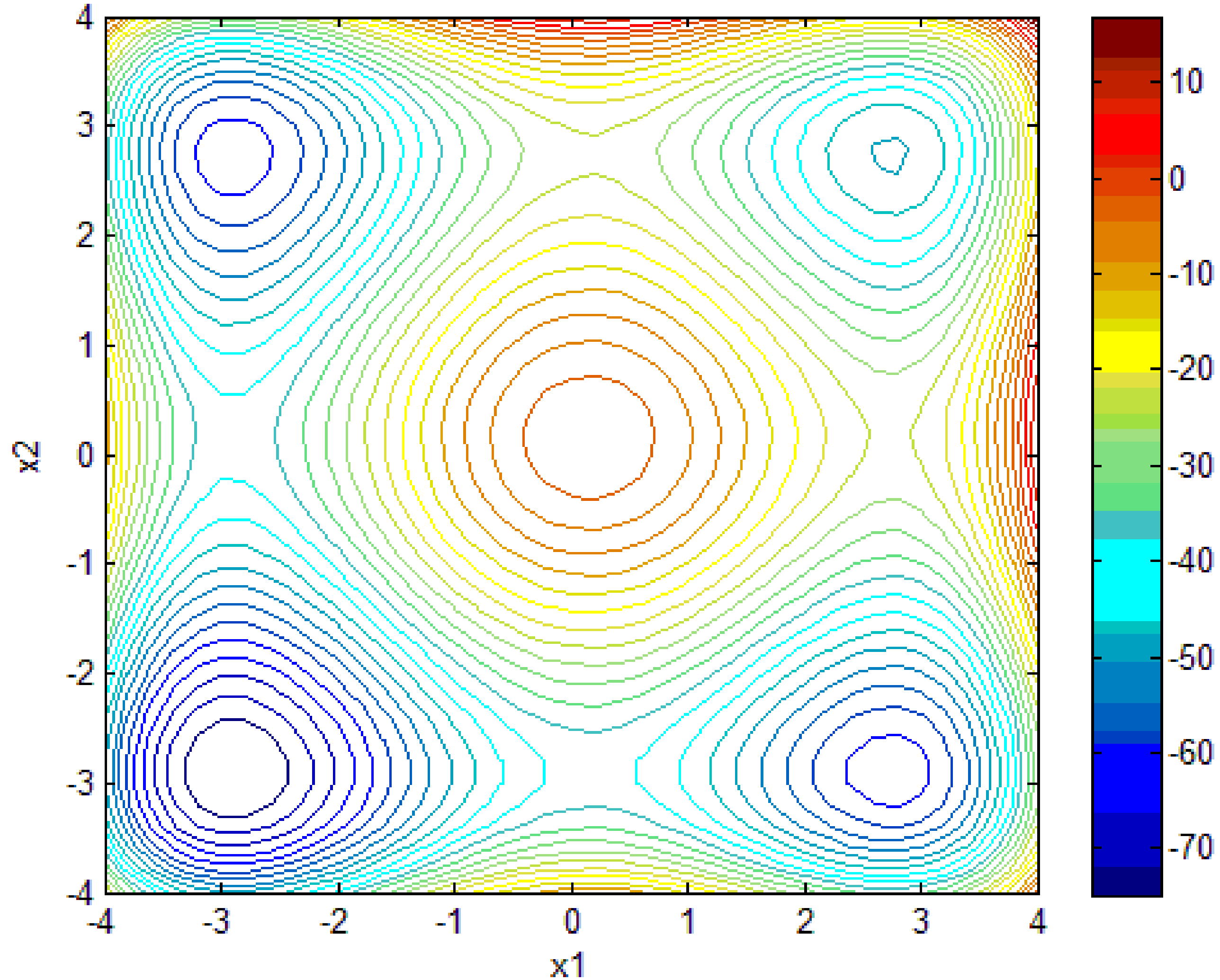
0.5*(4*x^3-32*x+5)= 0 dan 0.5*(4*y^3-32*y+5)= 0
y = 0.5*(4*x^3-32*x+5)
x = 0.5*(4*y^3-32*y+5)
25
20
15
10
5
0
-5
-10
-15
-20
-25
y
-25
-20
-15
-10
-5
0
5
10
15
20
25
x

Hessian matrix : $H(x,y)=\begin{pmatrix} 6x^2-16 & 0 \\ 0 & 6y^2-16 \end{pmatrix}$

Stationary points	λ_1	λ_2	Type
$P_1(-2.90353,-2.90353)$	34.5829	34.5829	local minima
$P_2(-2.90353,0.156732)$	34.5829	-15.8526	saddle point
$P_3(-2.90353,2.7468)$	34.5829	29.2695	local minima
$P_4(0.156731,-2.90353)$	-15.8526	34.5829	saddle point
$P_5(0.156731,0.156731)$	-15.8526	-15.8526	local maxima
$P_6(0.156731,2.7468)$	-15.8526	29.2695	saddle point
$P_7(2.7468,-2.90353)$	29.2695	34.5829	local minima
$P_8(2.7468,0.156732)$	29.2695	-15.8526	saddle point
$P_9(2.7468,2.7468)$	29.2695	29.2695	local minima



20
0
-20
-40
-60
-80
z
4
2
0
-2
-4
x2
-4
-2
0
2
4
x1



4
3
2
1
0
-1
-2
-3
-4
x2
-4
-3
-2
-1
0
1
2
3
4
x1
10
0
-10
-20
-30
-40
-50
-60
-70

Conclusions

Conclusions

- The root finding problem has been transformed to global optimization problem.
- Using a combination of the proposed clustering technique and spiral optimization algorithm, the roots of the systems of nonlinear equations can be localized and identified in a single run.
- Results from various test cases indicate that all real roots can be found without *a priori* knowledge of the number of the roots.
- To improve effectiveness, especially for *n*-D problem, setting the parameters of the algorithm must be carefully done.

Conclusions

- A modified Spiral Optimization Algorithm (m-SOA) may be used to solve MINLP problems.

- The use of Sobol sequence of points, instead of pseudo random points, to generate initial population of points for the SOA may enhanced the effectiveness of the method to obtain the optimal solution. .

- Combination of the proposed clustering technique and SOA have been shown able to obtain the maximum and minimum points (both local and global) of the multimodal functions in a single run.

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Thank you for your attention