Commutative Rings in General and the last development of my Research and Further Development

1. Commutative Rings

• $\mathbb{Z}$ is the ring of integers and

• $\mathbb{Q}$ is the field of rationals

• $\mathbb{Z} \ni a > 0$, (1) $a = p_1^{e_1} \dots p_r^{e_r}$, a finite products of prime numbers $p_i \dots (*)$.

(2) (**the uniqueness**) If $a = q_1^{f_1} \dots q_s^{f_s}$ is another decomposition of a, where q_i are prime numbers, then

$r = s, p_i = q_i$ and $e_i = f_i$ for all $i(1 \leq i \leq r)$.

•

$$\begin{array}{ccl} & & \mathbb{Q}(\sqrt{-5}) = \{a + \sqrt{-5}b \mid a, b \in \mathbb{Q}\}, \text{ a field} \\ & \diagup & \quad\quad\quad | \\ \mathbb{Q} & & \mathbb{Z}(\sqrt{-5}) = \{a + \sqrt{-5}b \mid a, b \in \mathbb{Z}\}, \text{ a ring.} \\ | & \diagup & \\ \mathbb{Z} & & \end{array}$$

Consider

$$21 = 3 \times 7 = (4 + \sqrt{-5})(4 - \sqrt{-5})$$

$3, 7, 4 + \sqrt{-5}, 4 - \sqrt{-5}$ are all prime elements in

$D = \mathbb{Z}(\sqrt{-5})$, which means the uniqueness does not hold.

Let consider i"deals" in stead of "(prime)elements" as follows::

$P_1 = 3D + (4 + \sqrt{-5})D, P_2 = 3D + (4 - \sqrt{-5}D,$

$P_3 = 7D + (4 + \sqrt{-5})D$ and $P_4 = 7D + (4 - \sqrt{-5})D$.

Definition. an ideal P of a ring is called **prime** if $ab \in P$, then either $a \in P$ or $b \in P$.

In fact, $P_1, P_2, P_3.P_4$ are all prime ideals of D.

We have $3D = P_1P_2$ and $7D = P_3P_4$ and

Hence $21D = 3D \times 7D = P_1P_2P_3P_4$ and the decomposition of $21D$ to prime ideals is unique.
Dedekind recovered the uniqueness by using ideals. This was the starting point of ideal theory.

In the reminder of this section, for simplicity, let D **be a commutative integral domain with its quotient filed** K**.**

• Let D and D' be domains and $f : D \Longrightarrow D'$ be a homomorphism from D to D'. Then Ker f $= \{ r \in D \mid f(r) = 0 \}$ is an ideal of D and $D/Kerf \cong f(D')$.

• Let P be a prime ideal of D and $\mathcal{C} = \{r \in D \mid r \notin P\}$, which is a multiplicatively closed set.

$D_P = \{rc^{-1} \mid r \in D$ and $c \in \mathcal{C}\}$, **the localization of** D **at** P

• $D = \cap D_P$, where P runs over all maximal ideals of D.

• $\mathbb{Z} = \cap \mathbb{Z}_P$, where $P = p\mathbb{Z}$ and p are prime numbers. Further, the chains: $\mathbb{Z}_P \supset p\mathbb{Z}_P \supset p^2\mathbb{Z} \supset \cdots \supset p^n\mathbb{Z}_P \supset \ldots,$, which are all non-zero ideals of $\mathbb{Z}_P$ and $\cap p^n\mathbb{Z}_P = (0)$.

Definition. A commutative domain V with its quotient field K is called a **valuation domain** if for any $k \in K$ either $k \in D$ or $k^{-1} \in D \Longleftrightarrow$
the set of all ideals V is linearly ordered by inclusions, that is, for any ideals $\mathfrak{a}$ and $\mathfrak{b}$, either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{a} \supseteq \mathfrak{b}$.

In particular, a valuation ring V is **discrete rank one** if $\cap \mathfrak{a}^n = (0)$ for any non-zero ideal $\mathfrak{a}$.
Hence $\mathbb{Z}_P$ is a discrete rank one valuation domain.

• Let $\mathbb{Q} \ni q = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ with $b \neq 0$. Then $qa = b \in \mathbb{Z}$.

By using this property, we can extend the concept of ideals to D-submodule of K as follow:

Definition. A D-submodule A of K is called a **(frac-**

tional) D**-ideal in** K if
$aA \subseteq D$ for some $0 \neq a \in D$.

Theorem 1.1 The following conditions of D are equivalent:
(1) Any ideal of D is a finite product of prime ideals.
(2) The set of all fractional D-ideals $G(D)$ is a group under the usual ideal product (note: D is the identity in $G(D)$).
(3) Any non-zero ideal of D is invertible.
(4) (i) $D = \cap D_P$, where P runs over all prime ideals of of D and
(ii) the set of all ideals of D_P is : $D_P \supset PD_P \supset P^n D_P \supset \cdots \supset (0)$.

Definition. An integral domain D is called **Dedekind** if one of the conditions in Theorem 1.1 is satisfied.

• $\mathbb{R}$ is a field of real numbers and $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}$ and $i^2 = -1\}$.

Definition. an element $k \in \mathbb{C}$ is called **integral over** $\mathbb{Z}$ if
there is a monic polynomial $f(x) = x^n + a_{n+1}x^{n-1} + \cdots + a_0 (a_i \in \mathbb{Z})$ such that $f(k) = 0$.

• consider

$$\begin{array}{ccc} & & K \subseteq \mathbb{C} \\ & / & | \\ \mathbb{Q} & & D = \{k \in K \mid k \text{ is integral over } \mathbb{Z} \ \} \\ | & / & \\ \mathbb{Z} & & \end{array}$$

, where K is a field with $\mathbb{Q} \subset K \subset \mathbb{C}$.

Theorem 1.2. If $[K : \mathbb{Q}] = n < \infty$, then D is a Dedekind domain.

• However if $[K : \mathbb{Q}] = \infty$, then D is not necessarily Dedekind domain.

Theorem 1.3. If $[K : \mathbb{Q}] = \infty$, then D is a (**Prüfer** domain, that is, any finitely generated ideal is invertible $\iff$ for any prime ideal P, D_P is a valuation domain.

• (Another property of $\mathbb{Z}$) $q \in \mathbb{Q}$, $0 \neq a \in \mathbb{Z}$: If $aq^n \in \mathbb{Z}$ for all $n > 0$, then $q \in \mathbb{Z}$.

Definition. An integral domain D with its quotient field K is called **completely integral closed in** K if for $q \in K$, $0 \neq a \in D$ such that $aq^n \in D$ for all $n > 0$, then $q \in D$.

• We can characterize the concept of " completely integral closed" by using "ideals":

For any ideal A, $O(A) = \{q \in K \mid qA \subseteq A \ \}$, which is called an **order** of A.

Proposition1.4. D is completely closed $\Longleftrightarrow O(A) = D$ for any ideal A of D.

Proof. $\Longrightarrow$. Let A be an ideal of D. It is clear $O(A) \supseteq D$. Conversely, let $q \in O(A)$, that is, $Aq \subseteq A$ and so $Aq^2 = Aqq \subseteq Aq \subseteq A$. Continuing this process, we have $Aq^n \subseteq A \subseteq D$. So for all $n > 0$. for any nonzero $a \in A$, $aq^n \in D$. Hence $q \in D$, that is. $O(A) = D$.
$\Longleftarrow$ Suppose $aq^n \in D$ for all $n > 0$, where $q \in K$ and $D \ni a \neq 0$. Then $A = a\{d_n + q^n + d_{n-1}q^{n-1} + \ldots d_0 \mid d_i \in D, n \geq 0\ \}$ is an ideal of D with $Aq \subseteq A$. Hence $q \in O(A) = D$ and so D is completely integral closed.

Definition. An integral domain D is called a**maximal order** if D is satisfied the conditions in Proposition 1.4.

- **Polynomial rings**

$D[x] = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \mid a_i \in D\ \}$, the polynomial ring over D in variable x.
In case $D = K$, it is well known that $K[x]$ is a principal ideal ring so that $K[x]$ is Dedekind.

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In case D is a Dedekind domain, then $D[x]$ is not Dedekind if $D \neq K$. In fact, $D[x]$ is a Krull domain.

The following diagram is a classification of integral domains from the view-point of arithmetic ideal theory:

(DVDs) $\Rightarrow$ (PIDs) $\Rightarrow$(Dedekind domains)$\Rightarrow$(Pr$\ddot{u}$fer domains)

$\Downarrow$ $\Downarrow$

(Krull domians) $\Rightarrow$(v-Pr$\ddot{u}$er)

(DVDs)$\Rightarrow$(Valuation domains) $\Rightarrow$(Pr$\ddot{u}$fer).

Here "DVDs" is an abbreviation of " Discrete rank one valuation domains" and "PIDs" is an abbreviation of "Principal ideal domains".

2. Noncommutative rings

• In 1843, Hamilton found the following noncommutative rings:

$D = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1, ij = k, jk = i$ and $ji = -k, kj = -i, ik = -j\ \}$.

For any element $a = a_0 + a_1 i + a_2 j + a_3 k$,

$a(\frac{a_0}{b} - \frac{a_1}{b} i - \frac{a_2}{b} j - \frac{a_3}{b} k) = 1,$

where $b = a_0^2 + a_1^2 + a_2^2 + a_3^2 \neq 0$.

• A noncommutative ring D is called a **division ring** or **skew field** if
for any nonzero element $a \in D$ there is an element $a^{-1} \in D$ such that $aa^{-1} = 1 = a^{-1}a$..

• (the Wedderburn structure theorem)
A ring S is a simple artinian ring $\Longleftrightarrow S = M_n(D)$ for some division ring D and $n \geq 1$.

In the remainder of my talk, R is always a **prime Goldie ring**, that is,
R has quotient ring $Q(R) = M_n(D)$, where D is a division ring .

• $Q(R) = \{ac^{-1} \mid a, c \in R$ with c is regular $\}$.

• **Examples of noncommutative rings**.

1. $M_n(R)$**, the** $n \times n$ **matrix ring over a ring** R**,** $n \geq 1$ **and any subrings of** $M_n(R)$.

2. Algebras over commutative ring D.

Let D be an integral domain with its quotient field K:

$$\begin{array}{ccc} & Q = M_n(\Delta), \text{where } \Delta \text{ is a division ring} & \\ \diagup & \mid & \\ K & & R = \{q \in Q \mid q \text{ is integral over } D\} \\ \mid & \diagup & \\ D & & \end{array}$$

, where $K = \{q \in Q \mid qs = sq$ for all $s \in Q$ $\}$, **the center** of Q and D is the center of R. Then R is a noncommutative ring.

3.Rings of Morita Contexts

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix} \tag{1}$$

,where R, S are rings,V is an (R, S)bi-module and W is an (S, R) bi-module satisfying $VW \subseteq R$ and $WV \subseteq S$.

4. Noncommutative polynomial rings

Let σ be an automorphism of R and δ be a left σ-derivation of R, that is, (i) δ is an additive map from R to R, and
(ii) the multiplication is given by;

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b \text{ for all } a, b \in R.$$

• $S = R[x; \sigma, \delta] = \{f(x) = a_n x^n + \cdots + a_0 \mid a_i \in R\ \}$, the set of all polynomial over R, is a ring by the following definition

$$xa = \sigma(a)x + \delta(a) \text{ for any } a \in R.$$

• S is called an **Ore extension of** R.

In 1933, Ore defined the noncommutative polynomial ring and obtained that S is a principal ideal ring in case $R = D$ is a division ring.

• In case $\delta = 0$, $S = R[x; \sigma, 0]$ is called a **skew polynomial ring**.

• In case $\sigma = 1$, $S = R[x; 1, \delta]$ is called a **differential polynomial ring.**

5. Noncommutative Rees rings

• Let X be an invertible ideals of R with $\sigma(X) = X$ and let $T = R[t; \sigma, \delta]$ be an Ore extension of R in variable t.

$T \supset S = R[Xt; \sigma, \delta] = R \oplus Xt \oplus \cdots \oplus X^n t^n \oplus \ldots$
is called an **Ore -Rees ring**.

• In case $\delta = 0$, then $S = R[Xt; \sigma, 0]$ is called a **skew Rees ring**

• In case $\sigma = 1$, $S = [Xt; 1, \delta]$ is called a **differential Rees ring**.

6. Graded rings of type G

Let G be any group and R be a ring.

• $R = \oplus_{g\in G} R_g$ is called a **graded ring of type** G if there are a family of additive subgroups $\{ R_g \}$ of R satisfying $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

• Graded rings contain group rings and crossed product algebras.

6. Some more examples of noncommutative rings

(i) PBW extensions (including universal enveloping algebras)

(ii) Hope Algebras

(iii) Quantum type algebras

(iv) formal power series rings and so on.

• Some differences between commutative rings and noncommutative rings

• In commutative domains, (invertible ideals) $\Longleftrightarrow$ (projective ideals).
In noncommutative rings, (invertible ideals) $\Longrightarrow$ (projective ideals). But the converse is not true in general.
• this difference induces three different concepts of commutative Dedekind domains, namely, noncommutative Dedekind rings, Hereditary rings and Asano rings.

$$T = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix} \tag{2}$$

$$|$$

$$R = \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix} \tag{3}$$

$$|$$

$$S = \begin{pmatrix} Z & pZ \\ Z & Z \end{pmatrix} \tag{4}$$

, where Q is the field of nationals , Z is the ring of integers and p is a prime number.

R is a Dedekind ring and S is a hereditary.

Let D be a Noetherian simple ring, e.g., the first Wely algebra with characteristic =0. Then

$D[x]$, the polynomial ring over D, an Asano ring.

Definitions

(1) R is called a **Dedekind ring** if
(i) R is a **maximal order** in $Q(R)$, that is, $O_l(A) = \{q \in Q(R) \mid qA \subseteq A\} = R = O_r(A) = \{q \in Q(R) \mid Aq \subseteq A\}$ for any (two-sided) ideal A.
(ii) any left(right)ideal is projective.

(2) R is called a **hereditary ring (an HNP ring for short**) if any left and right ideal is projective.

(3) R is called an **Asano ring** if any ideal is invertible.

• (Dedekind rings) $\Longrightarrow$ (Asano rings)
$\Downarrow$
(Hereditary rings)

• The following diagram is a classification of noncommutative rings from views-point of arithmetic ideal theory:

(Noetherian valuation rings) $\Rightarrow$(PIRs) $\Rightarrow$(Dedekind)$\Rightarrow$(Asano)
$\Downarrow$
$\Rightarrow$(Krull) (Heredidary)
$\Downarrow$
(Semiheridiary)

$\Uparrow$

(Valuation rings) $\Longrightarrow$ (Prüfer) $\Rightarrow$ (v-Prüfer)

3. Joint research works with Gadjah Mada University

• Let $S = R[x; \sigma]$, a skew polynomial rings or $S = R]x; \delta]$, a differential polynomial rings.

In case R is an HNP ring,
I have already describe all projective ideals (including a description of invertible ideals) in the following papers.

[1] Projective ideals of skew polynomial rings over HNP rings, Comm. in Algebra, 2017 (with E. Akalan, P. Aydogdu, B.Sarac and A. Ueda).

[2] Projective ideals of differential polynomial rings over HNP rings, to appear in the Proceedings, Algebra and its Applications to be published by De Gruter, Germany.(with A.Ueda).

Theorem 3.1. any projective ideal of S is of the form:

$$X\mathfrak{a}[x, \sigma] \ (\ X\mathfrak{b}[x, \delta])$$

where X is an invertible ideal of S and $\mathfrak{a}$ is a σ-invariant ideal of R($\mathfrak{b}$ is aδ-stable ideal of R).

• (1) Let $S = R[Xt; \sigma]$ be a skew Rees ring or $S = R[xt; \delta]$ be a differential Rees rings.

• **Question1. Describe all projective ideals of** S.

Propostion 3.2. (i) Let A be an ideal of S with $A \cap R \neq (0)$. Then A is a projective ideal $\Longleftrightarrow A = \mathfrak{a}[Xt; \sigma]A = (\mathfrak{b}[Xt; \delta])$, where $\mathfrak{a}$ is a σ-invariant ideal of R ($\mathfrak{b}$is a δ-stable ideal of R.
(ii) If $A \cap R = (0)$, then the question is still open.

• I got the following two paper from Gadjah Mada side:

.1. Sutopo, Indah Wijayanti and Sri Wahyuni, Some results on Grade N-prime submodules, Far East J. Math, 2016.
2. ————, On graded N=prime submodule of a graded frac-

tion module,Far East J. Math., 2017,

• F. Van Oystaeyen and A. Verschren: Relative Invariants of Rings, Marcel Dekker, INC, 1984. They studied graded rings from the point-view of arithmetic ideal theory.

• Let $R = \oplus_{n \in Z} R_n$ be a strongly grade ring of type $\mathbb{Z}$ and R_0 be a prime Goldie ring with its quotient ring Q_0 and let $\mathcal{C}$ be the set of all regular elements of R_0.Then $\mathcal{C}$ is an Ore set of R and

$Q^g = \{rc^{-1} \mid r, c \in R \text{ and } c \in \mathcal{C}\ \}$ is called a **graded quotient ring of R**.

F. Van Oystaeyen and A. Verschren obtained the following:

Theorem 1*. If R_0 is a maximal order,then so is R.

Theorem 2*. If R_0 is a Krull order, then so is R.

Proposition 3*.The following conditions are equivalent:

(i) R is a graded Asano order
(ii) R is a graded maximal order in Q^g and every nonzero graded ideal is projective, that is, left and right projective.
(iii) every graded ideal is invertible.

Definition R is a **graded maximal order in** Q^g if $O_l(A) = R = O_r(A)$ for any graded ideal of A of R.

• In their Theorems 1* and 2*, they do not mention of the converse and in Proposition 3*, they do not mention of R_0.

Definition. (1) An ideal A_0 of R_0 is called $\mathbb{Z}-$**invariant** if $R_n A_0 = A_0 R_n$ for any $n \in \mathbb{Z}$.

(2) R_0 is called a **$\mathbb{Z}$- invariant maximal order** if $O_l(A_0) = R_0 = O_r(A_0)$ for any $\mathbb{Z}$ -invariant ideal A_0 of R_0.

• (maximal orders) $\Longrightarrow$ ($\mathbb{Z}$-invariant maximal order).

The converse is not necessarily to be held:

• **the counter example**: Let $R_0 = \begin{pmatrix} Z & pZ \\ Z & Z \end{pmatrix}$, Z is ring of integers. $X = \begin{pmatrix} pZ & pZ \\ Z & pZ \end{pmatrix}$ is invertible ideal of R_0. Put $R_n = X^n$ for any integers n.

$$R = \oplus R_n$$

is a $\mathbb{Z}$-invariant maximal order but R_0 is not a maximal order.

• By using the concept of **$\mathbb{Z}$-invariant maximal orders, $\mathbb{Z}$-invariant Asano orders** and **$\mathbb{Z}$-invariant Krull order**. We can improve a part of Oystaeyen and Verschron's book.

Theorem 3.1. R_0 is a $\mathbb{Z}$-invariant maximal order in $Q_0 \Longleftrightarrow R$ is a maximal order $Q(R)$.

Theorem 3.2. R_0 is a $\mathbb{Z}$-invariant Krull order in $Q_0 \Longleftrightarrow R$ is a Krull order in $Q(R)$(this is not complete yet)

Proposition 3.3. In addition to the conditions (i), (ii), (iii) in Proposition 3*
(iv) R_0 is a $\mathbb{Z}$-invariant Asano order in Q_0, that is, any $\mathbb{Z}$-invariant ideal is invertible.
(v) R is an Asano order in $Q(R)$.

the conditions (i) $\sim$ (v) are equivalent?

• The following projects are joint research work in future (?).

Let $R = \oplus_{n \in Z} R_n$ be a strongly graded ring of type $\mathbb{Z}$.

Question 2. Describe all projective ideals of R in case R_0 is an HNP ring.

• Let $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a ring of Morita context. $R = \oplus_{n \in Z} R_n$, $S = \oplus_{n \in Z} R_n$ are graded ring of type $\mathbb{Z}$, $V = \oplus_{n \in Z} V_n$ is a graded (R, S)-bimodule and $W = \oplus_{n \in Z} W_n$ is a graded (S, R)-bimodule..

Question 3. Study the structure of (graded) ideal theory in T.