PERAMBATAN SOLITON PADA MEDIUM NONLINEAR KERR NONHOMOGEN

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Persamaan NLS pada Medium Nonhomogen

Beam propagation equation in nonlinear Kerr media with a transverse linear refractive index variation, i.e.

$$n_0 \to n_0\left(1+\Delta n(X)\right)$$

$$\text{Maxwell's equations} \longrightarrow \text{NLH}$$

$$E = \kappa\sqrt{n_0 / n_2}\, A \exp(ik_0 z)$$

$$\Delta\tilde{n} = \Delta n / \kappa^2$$

$$\underbrace{i\frac{\partial A}{\partial Z} + \frac{1}{2}\frac{\partial^2 A}{\partial X^2} + |A|^2 A + \Delta\tilde{n} A}_{\text{SVEA}} + \cancel{\kappa^2\left(\frac{1}{2}\frac{\partial^2 A}{\partial Z^2} + \frac{1}{2}\Delta\tilde{n} A + \Delta\tilde{n}|A|^2 A\right)} = 0.$$

MODEL MATEMATIKA

- Medium Kerr lokal homogen ➔ persamaan NLS:

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2 u=0$$

- Medium Kerr nonhomogen - nonlinear

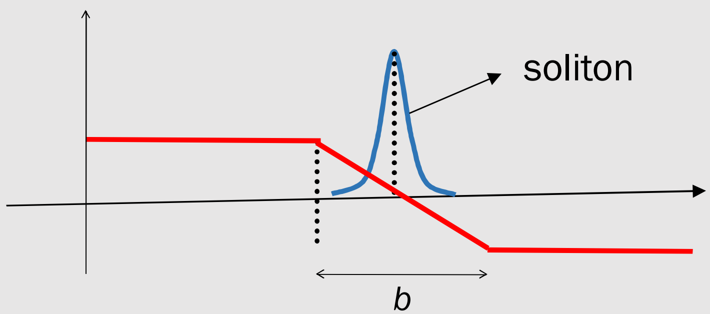
  ➔ persamaan m-NLS

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2 u=Vu,$$

$$V=-\Delta n(x)$$

$$\Delta n(x)=\begin{cases}\dfrac{\Delta n_0}{2}, & x<-\dfrac{b}{2}\\ -\dfrac{\Delta n_0}{b}x, & -\dfrac{b}{2}\le x\le\dfrac{b}{2}\\ -\dfrac{\Delta n_0}{2}, & x>\dfrac{b}{2}\end{cases}$$

$\Delta n(x)$ (Linear refractive index profile)



PENDEKATAN PARTIKEL - EKIVALEN

➔Soliton diasumsikan sebagai partikel tunggal yang tidak pecah dan didefinisikan:

- energi: $p(z) = \int_{-\infty}^{\infty} |u|^2 \, dx$
- pusat berkas: $\bar{x}(z) = \dfrac{\int_{-\infty}^{\infty} x |u|^2 \, dx}{p(z)}$
- kecepatan: $v(z) = \dfrac{1}{2} i \dfrac{\int_{-\infty}^{\infty} u \partial_x u^* - u^* \partial_x u dx}{p(z)}$

From m-NLS equation ➔

$$\frac{dp}{dz} = 0; \frac{d\bar{x}}{dz} = v(z) \text{ dan}$$

$$a(\bar{x}) = \frac{d^2 \bar{x}(z)}{dz^2} = -\frac{\partial U(\bar{x})}{\partial \bar{x}} \Leftarrow \quad \frac{dv}{dz} = \frac{d^2 \bar{x}}{dz^2} = -p^{-1} \int_{-\infty}^{\infty} \frac{dV}{dx} |u|^2 \, dx.$$

PENDEKATAN PARTIKEL - EKIVALEN

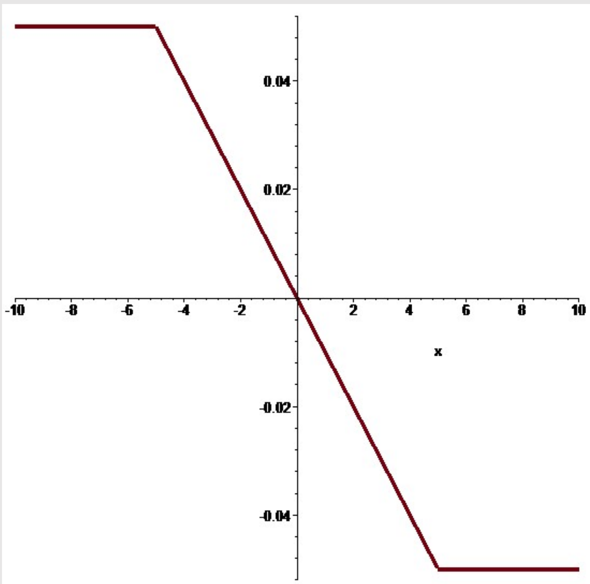
$$u(x,0) = q\,\mathrm{sech}\left(q\left(x - \bar{x}_0\right)\right)$$ Initial value

$$\Downarrow$$

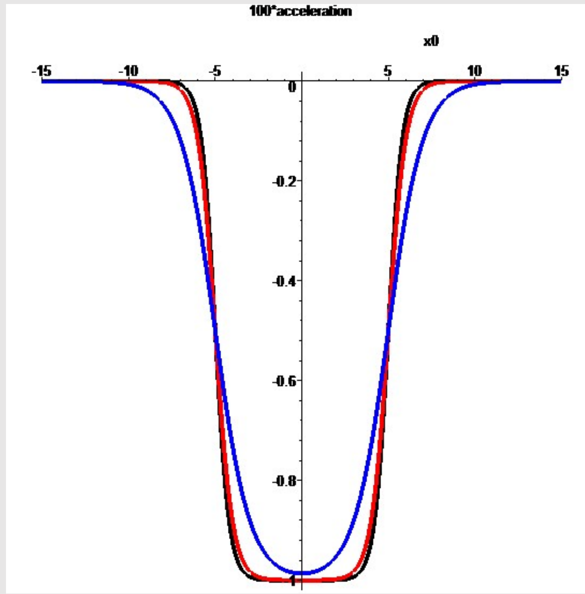
$$u(x,z) = q\,\mathrm{sech}\left(q\left(x - \bar{x}(z)\right)\right)\exp\left(i\left(v(z)x + \sigma(z)\right)\right)$$

$$a(\bar{x}) = -\frac{\Delta n_0}{2b}\frac{\sinh(qb)}{\cosh\left(\frac{1}{2}q(b-2\bar{x})\right)\cosh\left(\frac{1}{2}q(b+2\bar{x})\right)}$$

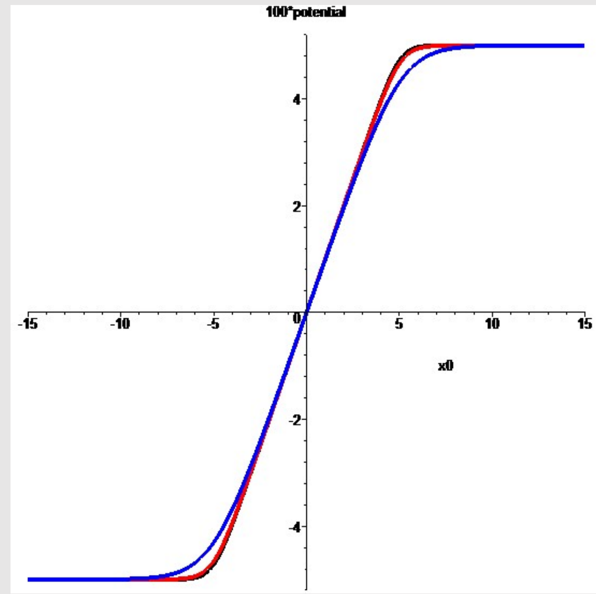
$$U(\bar{x}) = \frac{\Delta n_0}{2bq}\ln\left(\frac{\exp\left(-\frac{1}{2}qb\right)+\exp\left(\frac{1}{2}q(b+4\bar{x})\right)}{\exp\left(\frac{1}{2}qb\right)+\exp\left(-\frac{1}{2}q(b-4\bar{x})\right)}\right)$$



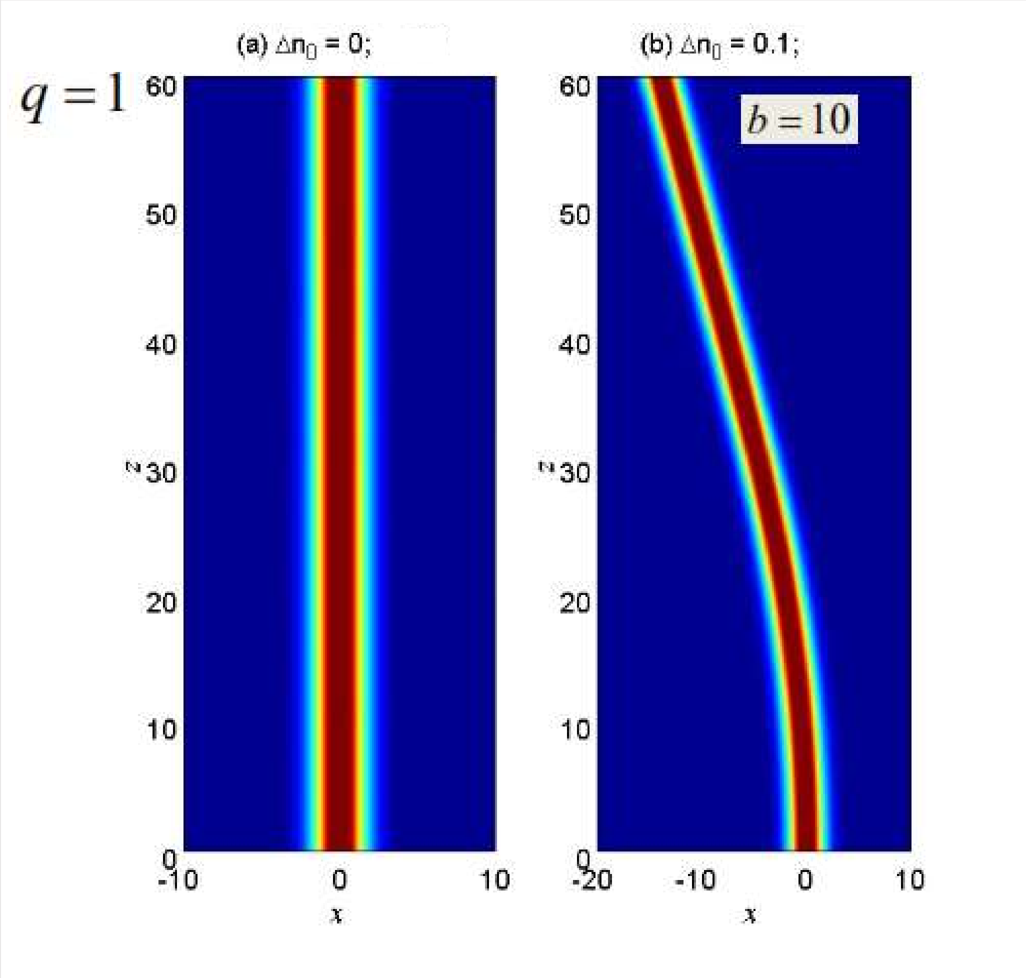
Profil Δn



Profil Percepatan (100x)



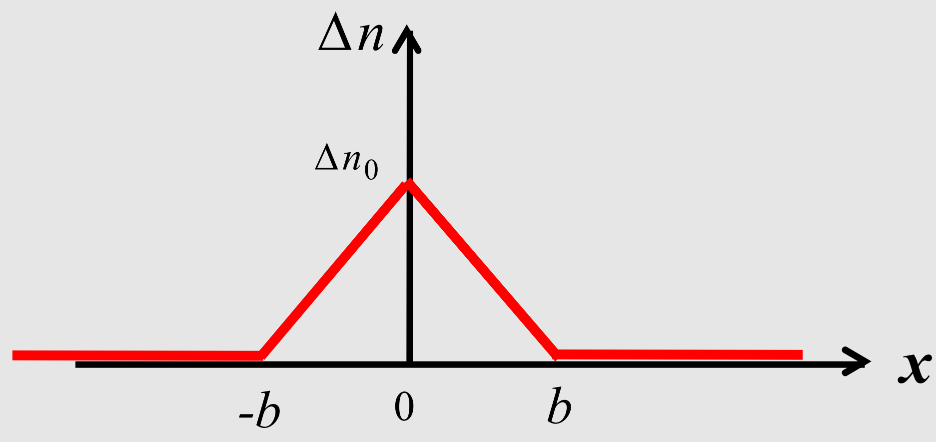
Profil Potential (100x)



(a) Δn0 = 0;
(b) Δn0 = 0.1;
q = 1
b = 10
z
x

Profil indeks bias linear berbentuk segitiga

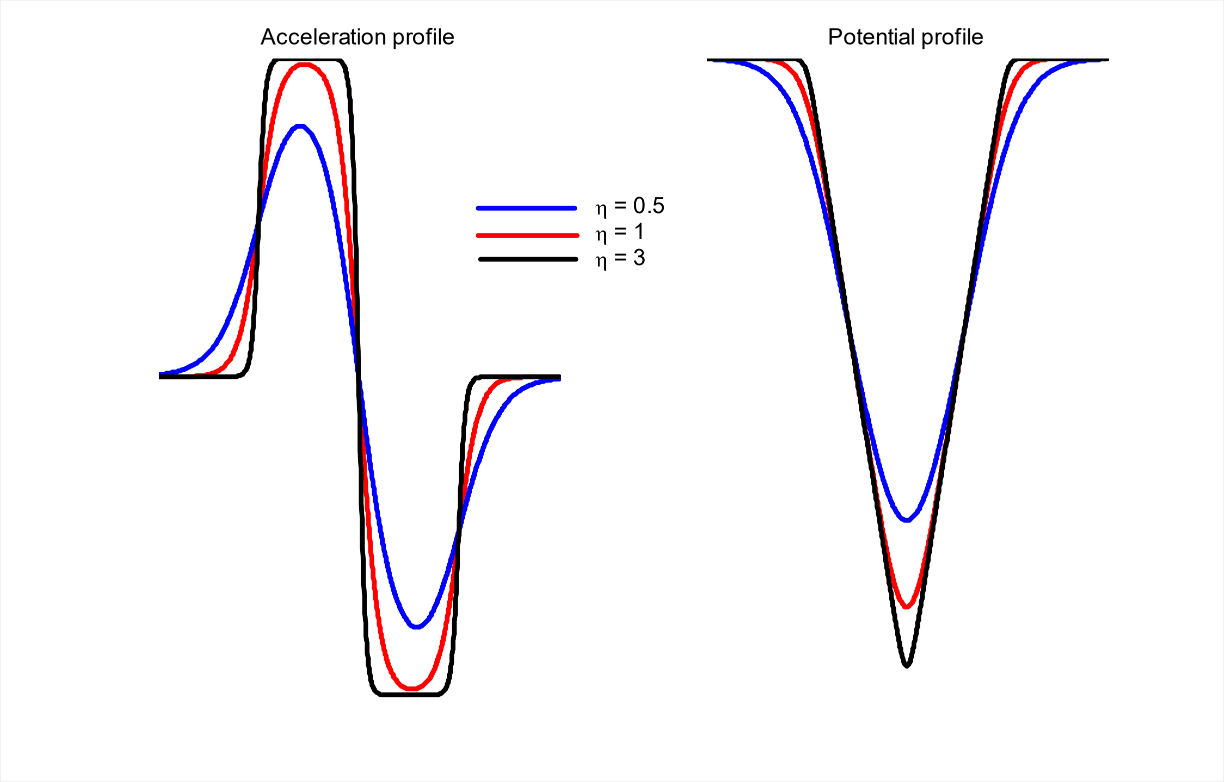
$$\Delta n(x) = \begin{cases} 0, & x < -b \\ \Delta n_0 \left(1 + \dfrac{x}{b}\right), & -b \le x < 0 \\ \Delta n_0 \left(1 - \dfrac{x}{b}\right), & 0 \le x < b \\ 0, & x \ge b \end{cases}$$



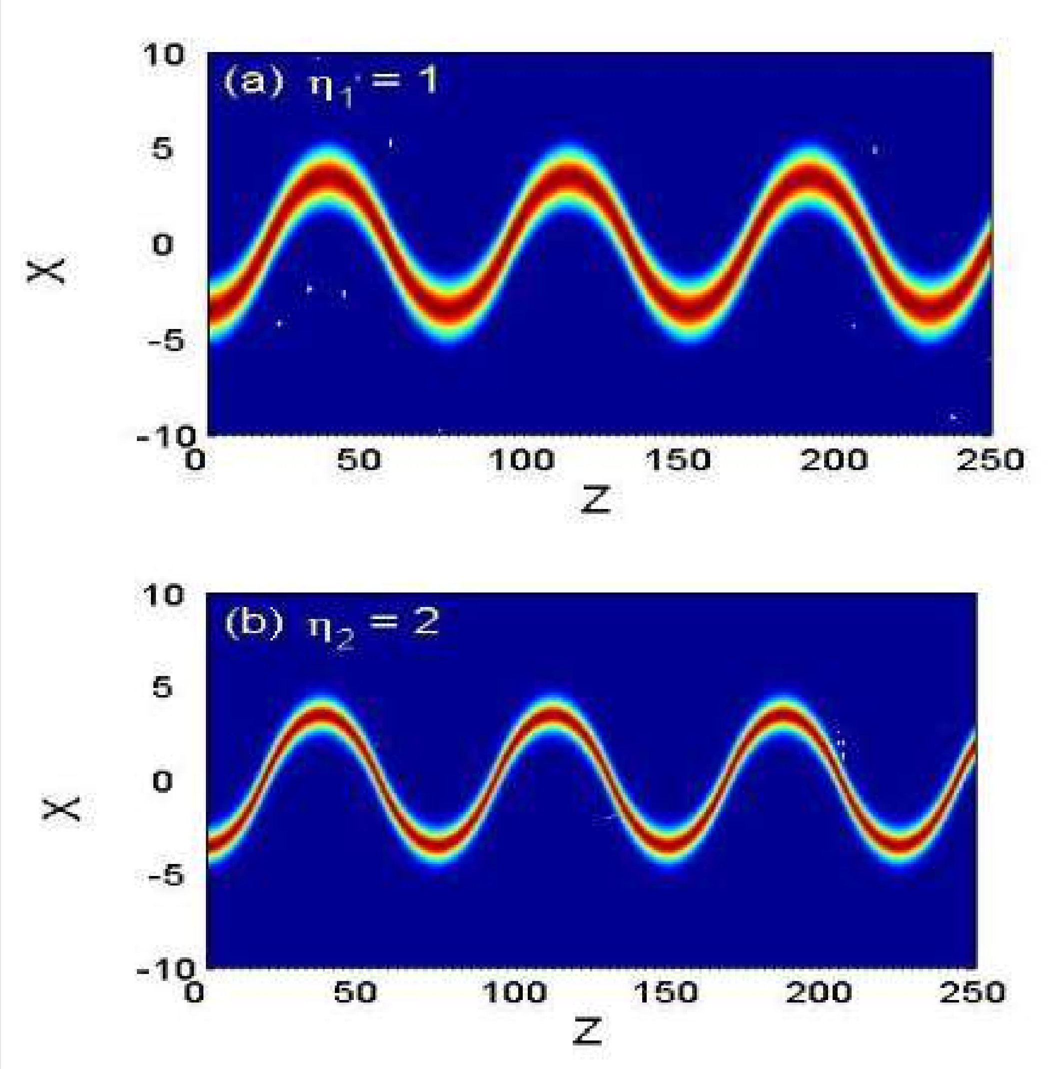
$$a\left(\bar{X}\right)=\left(\frac{\Delta n_0}{b}\right)\frac{e^{2q\bar{X}}\left(e^{2bq}-1\right)\left(e^{2bq}+e^{2b\bar{X}}-e^{2q\left(\bar{X}+b\right)}-1\right)}{e^{2q\bar{X}+1}\left(e^{2q\left(\bar{X}+b\right)}+1\right)\left(2bq+e^{2b\bar{X}}\right)}$$

$$U\left(\bar{X}\right)=-\frac{\Delta n_0}{2b}\ln\frac{\left(e^{2bq}+e^{2q\bar{X}}\right)\left(e^{2q\left(\bar{X}+b\right)}+1\right)}{\left(e^{2q\bar{X}}+1\right)^2}$$

For $u(x, z=0) = \eta\,\mathrm{sech}\left[\eta\left(x - x_0\right)\right]$:



$$u(x,0) = \eta \operatorname{sech}\left[\eta\left(x - x_0\right)\right]$$



Multi-soliton bound state

Nilai awal: $u(x,0) = Mq_0 \operatorname{sech}[q_0(x - \bar{x}_0)]$

Memuat *N* soliton dengan masing-masing amplitudo

$$q_j = 2q_0\left(M - j + \frac{1}{2}\right), \quad j = 1,2,\ldots,N$$

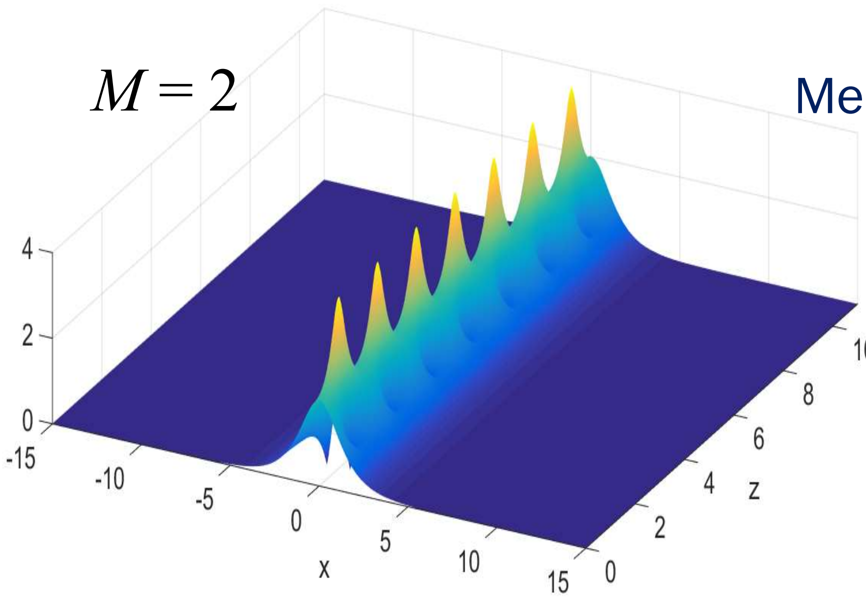
dengan

$$M - \frac{1}{2} < N \le M + \frac{1}{2}$$

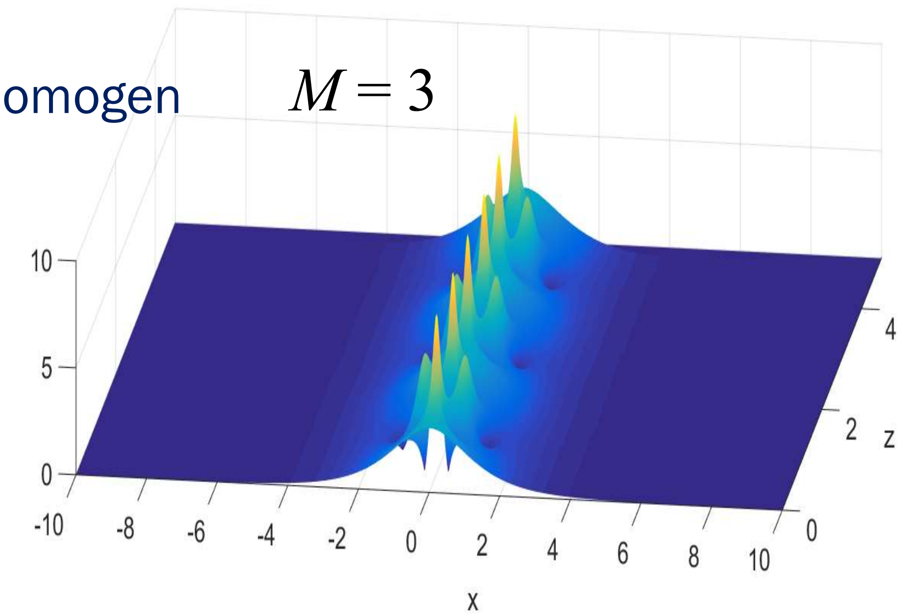
Nilai awal: $u(x,0) = M \operatorname{sech}(x)$

Medium homogen

$M = 2$

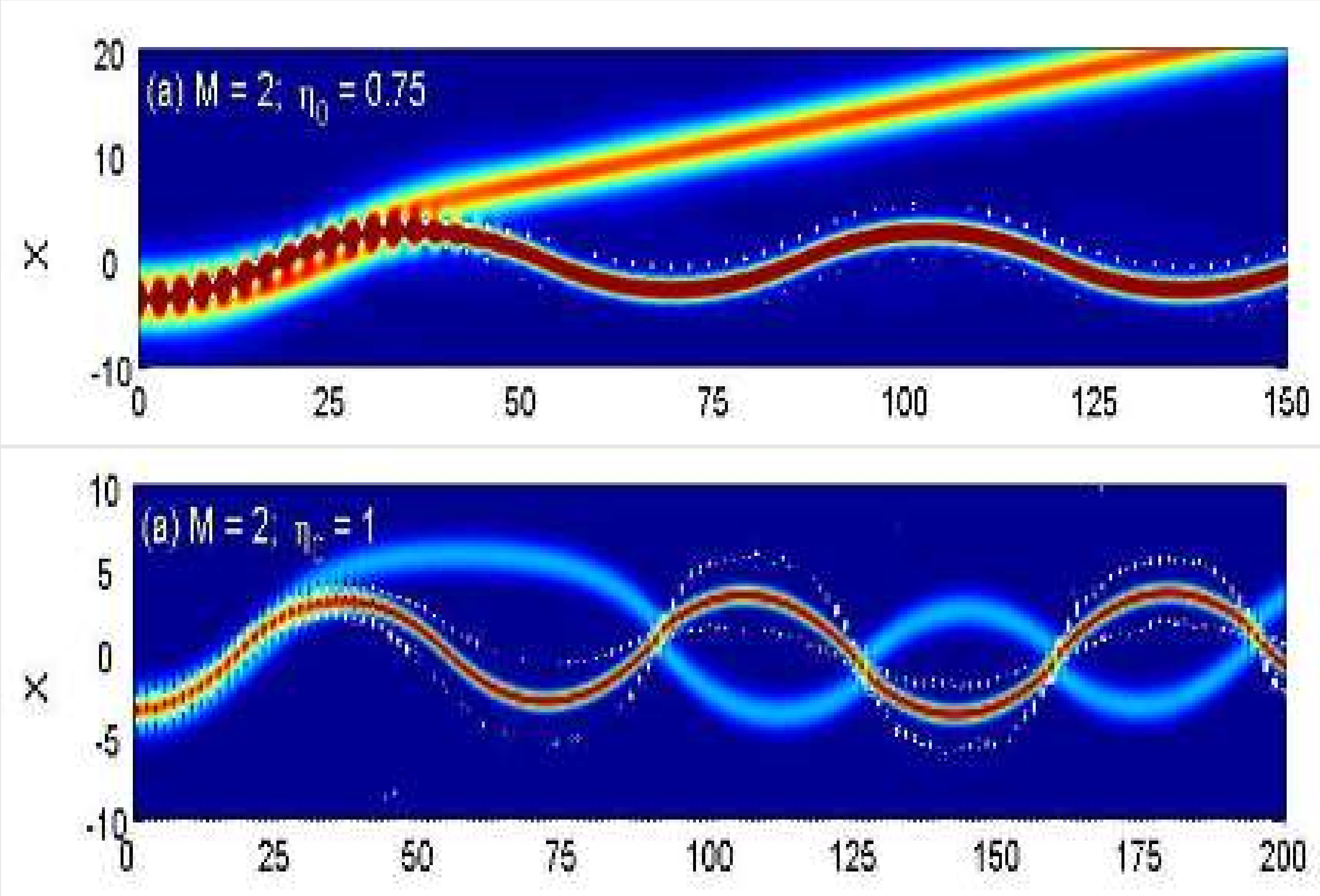


$M = 3$



Nilai awal: $u(x,0) = 2\eta_0 \operatorname{sech}(\eta_0 x)$

Medium nonhomogen (segitiga)



PERAMBATAN SOLITON PADA MEDIUM NONLINEAR KERR HOMOGEN

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KONSEP DASAR GELOMBANG:
Gelombang Monokromatik dan Relasi Dispersi

- Perhatikan persamaan evolusi linear berbentuk: $\partial_t u = Lu$

dengan L adalah operator diferensial linear dalam x dengan koefisien konstan. Untuk suatu polinom $p(\xi)$:

$$Lu = p(\partial_x)u.$$

Perhatikan bahwa jika $u = e^{ikx}$, maka $Lu = Le^{ikx} = p(ik)\, e^{ikx}$.

Catatan: Fungsi $p(\xi)$ disebut simbol dari operator diferensial L.

L disebut operator pseudo-diferensial jika fungsi $p(\xi)$ bukan berupa polinom.

- Perhatikan persamaan linear: $\partial_t u = Lu$

Ansatz: $u(x,t) = e^{i(kx-\omega\ )}$

dengan k = bilangan gelombang, ω = frekuensi dan $k = \frac{2\pi}{\lambda}$,

λ = panjang gelombang; sehingga

$$0 = \partial_t u - Lu = [-\mathrm{i}\omega - p(ik)]e^{i(kx-\omega t)}$$

$\omega = \Omega(k) \equiv ip(ik)$ ➡ Relasi Dispersi $k \mapsto \Omega(k), k \in \mathbb{R}, \omega \in \mathbb{C}$

Solusi: $u(x,t) = e^{i(kx-\Omega(k)t)}$ ➡ gelombang monokromatik

Jika $u(x,0) = u_0(x) = \int_{-\infty}^{\infty} \widehat{u_0}(k)e^{ikx}dk$

maka $u(x,t) = \int_{-\infty}^{\infty} \widehat{u_0}(k)e^{i(kx-\Omega(k)t)}dk$

Sifat solusi bergantung pada $\Omega(k)$

Persamaan Translasi sebagai gelombang sederhana

$$\partial_t u + c\partial_x u = 0$$

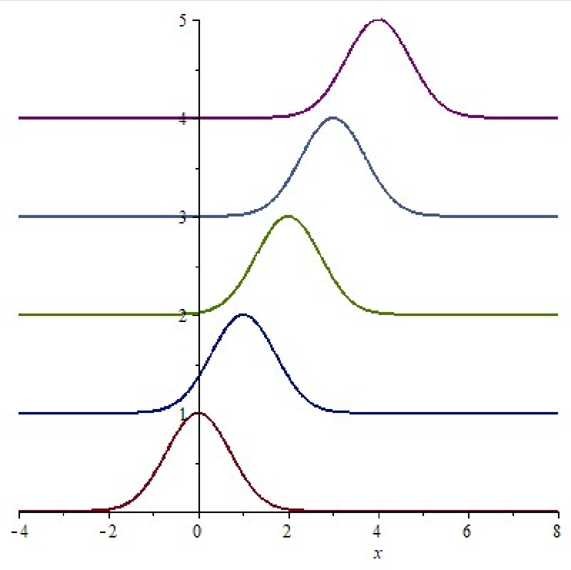
$$\Omega(k) = ck \quad \Rightarrow \quad e^{ik(x-ct)}$$

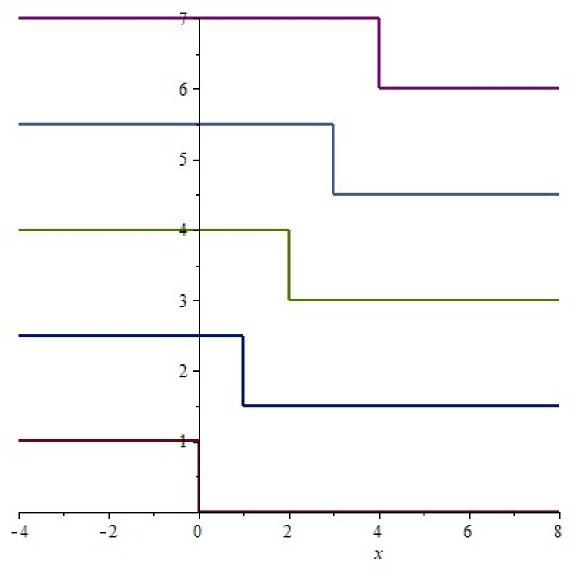
Jika $u(x,0) = f(x)$ maka $u(x,t) = f(x-ct)$.

c : kecepatan gelombang

$c > 0$ ➔ gelombang berjalan ke kanan

$c < 0$ ➔ gelombang berjalan ke kiri





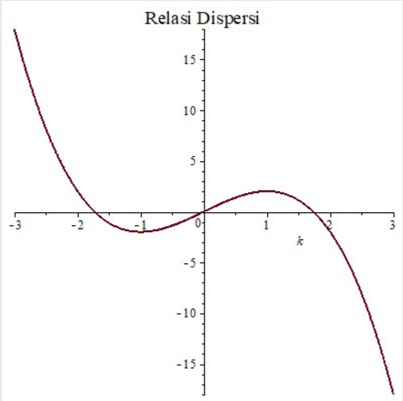
Dispersi

Gelombang monokromatik dengan bilangan/panjang gelombang berbeda merambat dengan kecepatan yang berbeda.

Contoh:

$$\partial_t u + 3\partial_x u + \partial_x^3 u = 0$$

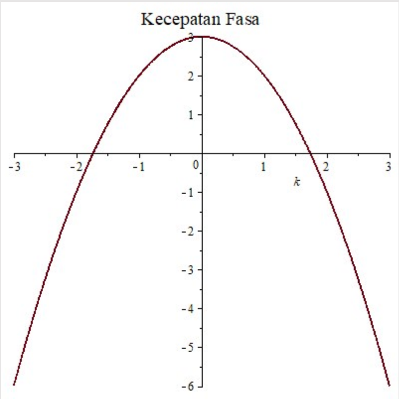
$$\Omega(k) = 3k - k^3$$

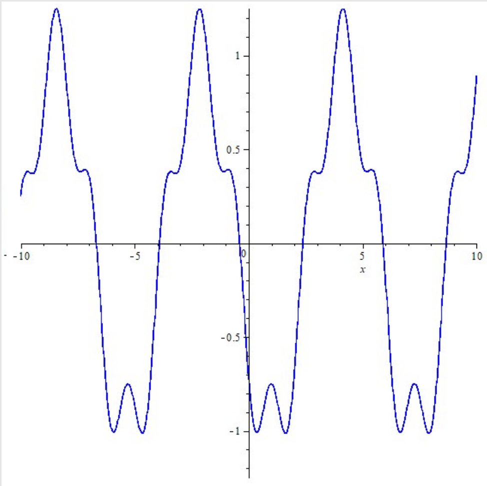


$$u(x,t) = \sin(kx - (3k - k^3)t)$$

Kecepatan Fasa

$$e^{i(kx - \Omega(k)t)} = e^{ik(x - V_{ph}t)}, V_{ph} = \Omega(k)/k$$





Gelombang Bikromatik

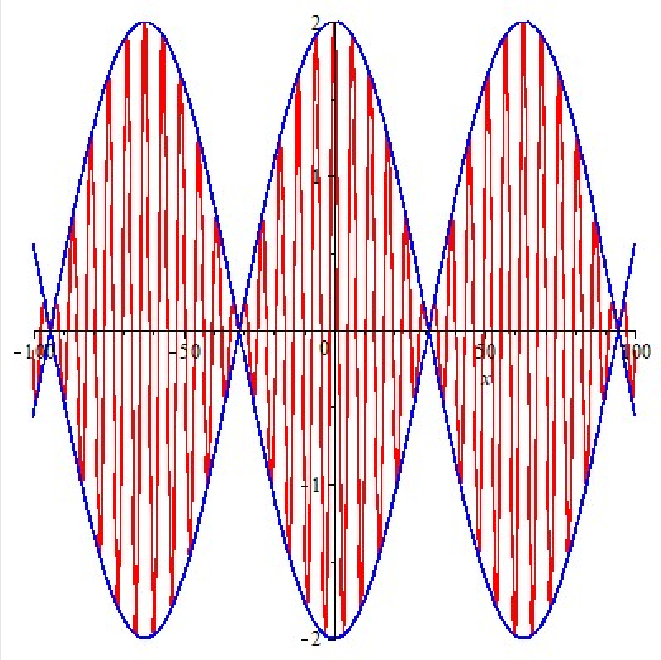
$$u(x,t) = e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)} = \overbrace{e^{i(k_0 x - \omega_0 t)}}^{\text{Carrier Wave}} \times \overbrace{2\cos\left(\Delta k\left(x - \frac{\Delta\omega}{\Delta k}\right)\right)}^{\text{Modulasi / Selubung gelombang}}$$

$$k_0 = \frac{1}{2}(k_1 + k_2);\ \ \omega_0 = \frac{1}{2}(\omega_1 + \omega_2);$$

$$\Delta k = \frac{1}{2}(k_1 - k_2);\ \ \Delta\omega = \frac{1}{2}(\omega_1 - \omega_2);$$

Contoh: $k_1 = 1;\ k_2 = 1.1;$
$\Omega(k) = k - 0.1k^3$

Kecepatan Grup: $V_{gr} = \frac{\Delta\omega}{\Delta k} \approx \frac{\partial\Omega(k_0)}{\partial k}$



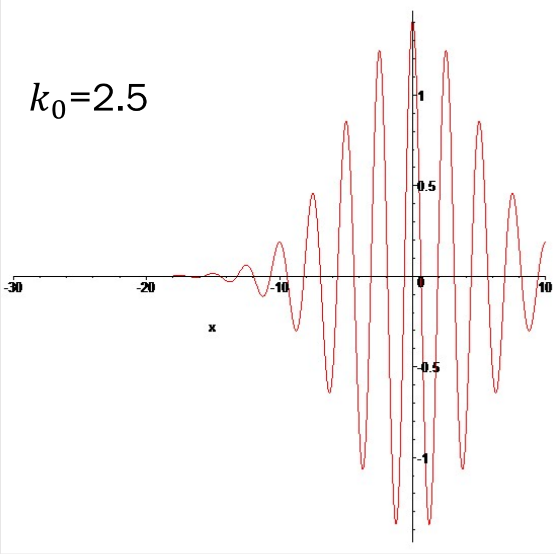
Wave Group

Misalkan fungsi spektral $\hat{f}$ berpuncak pada k_0 dan $V_{gr}(k_0) \neq 0$, sehingga

$$\Omega(k) = \Omega(k_0) + V_{gr}(k_0)(k - k_0) + \mathcal{O}((k - k_0)^2)$$

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{i(kx - \Omega(k)t)} dk \approx e^{i\left(\Omega(k_0) - k_0 V_{gr}(k_0)\right)t} f(x - V_{gr}(k_0)t)$$

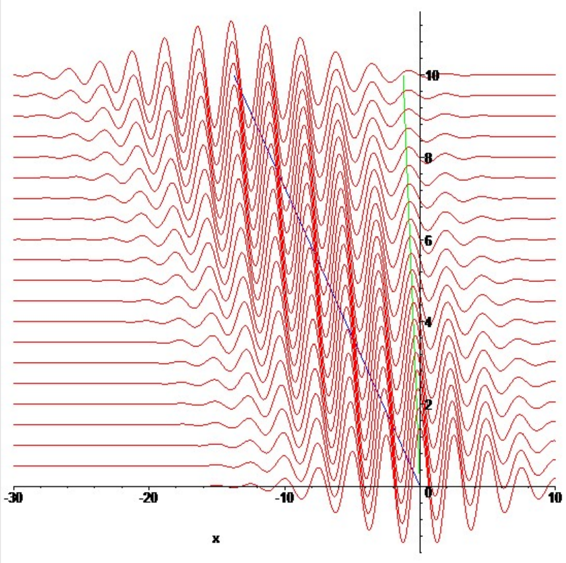
k_0=2.5



$$v := (x,t) \rightarrow \frac{1}{10}\left(\sum_{'m' = -25}^{25} \text{'}f\left(k0 + \frac{m}{10}\right)\cos\left(\left(k0 + \frac{m}{10}\right)x - \Omega\left(k0 + \frac{m}{10}\right)t\right)\text{'}\right)$$

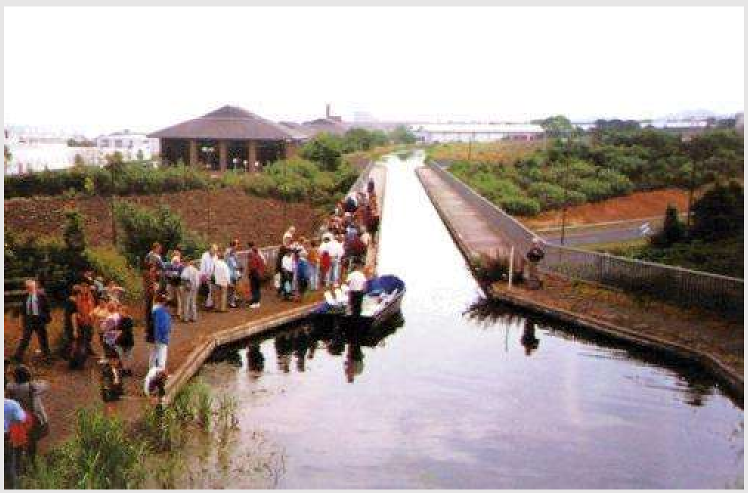
$$f := k \rightarrow \frac{\mathbf{e}^{\left(-\frac{1}{2}\frac{(k - k0)^2}{\sigma^2}\right)}}{\sigma\sqrt{\pi}}$$

$$\Omega := k \rightarrow k - 0.2\,k^3$$



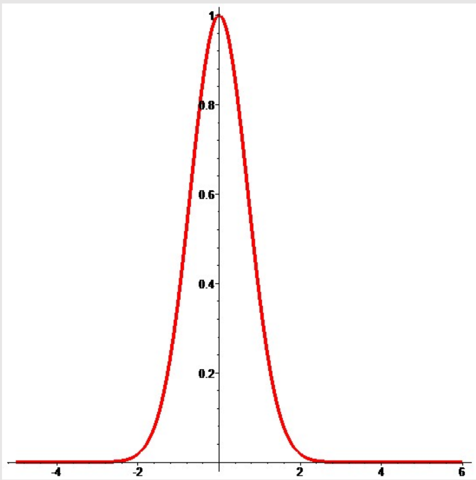
PERAMBATAN GELOMBANG OPTIK PADA MEDIUM NONLINIER KERR

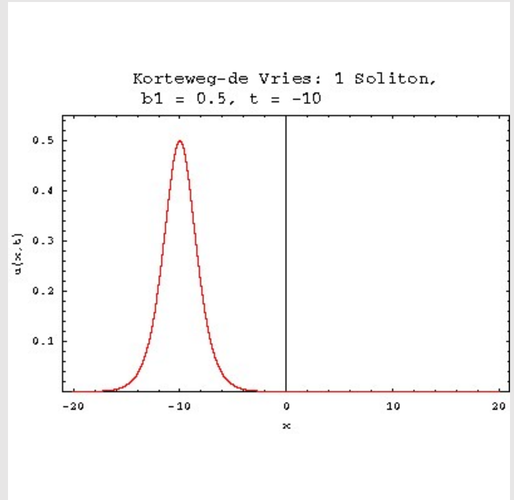
Recreation of the Wave of Translation (1995)

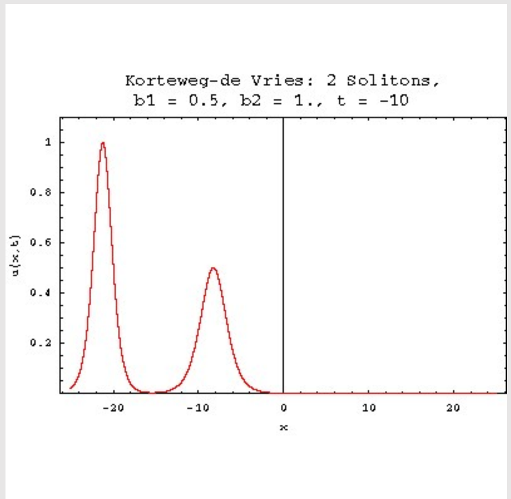


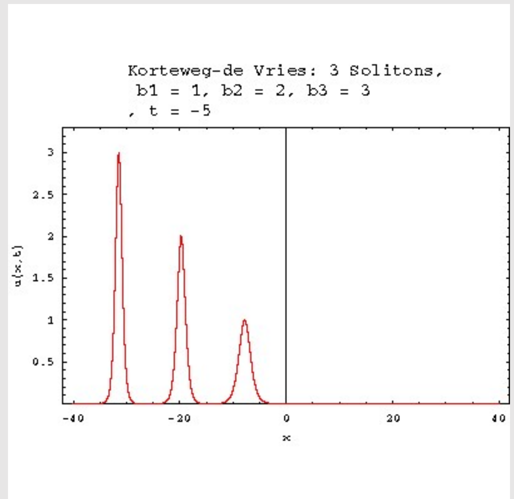
Scott Russell Aqueduct on the Union Canal
near Heriot-Watt University, 12 July 1995

$$\underbrace{u_t + uu_x}_{\text{Nonlinearitas}} + \underbrace{u_{xxx}}_{\text{Dispersi}} = 0$$

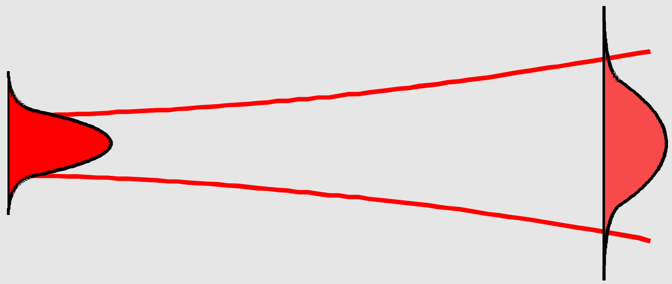






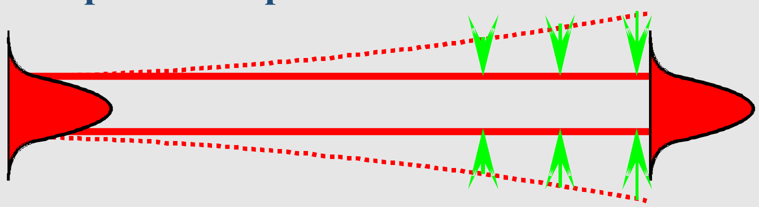


All Wave Phenomena: A Beam Spreads in Time and Space on Propagation



Space: Broadening by Diffraction
Time: Broadening by Group Velocity Dispersion

Spatial/Temporal Soliton



Broadening +
Narrowing Via **a Nonlinear** Effect
= Soliton (Self-Trapped beam)

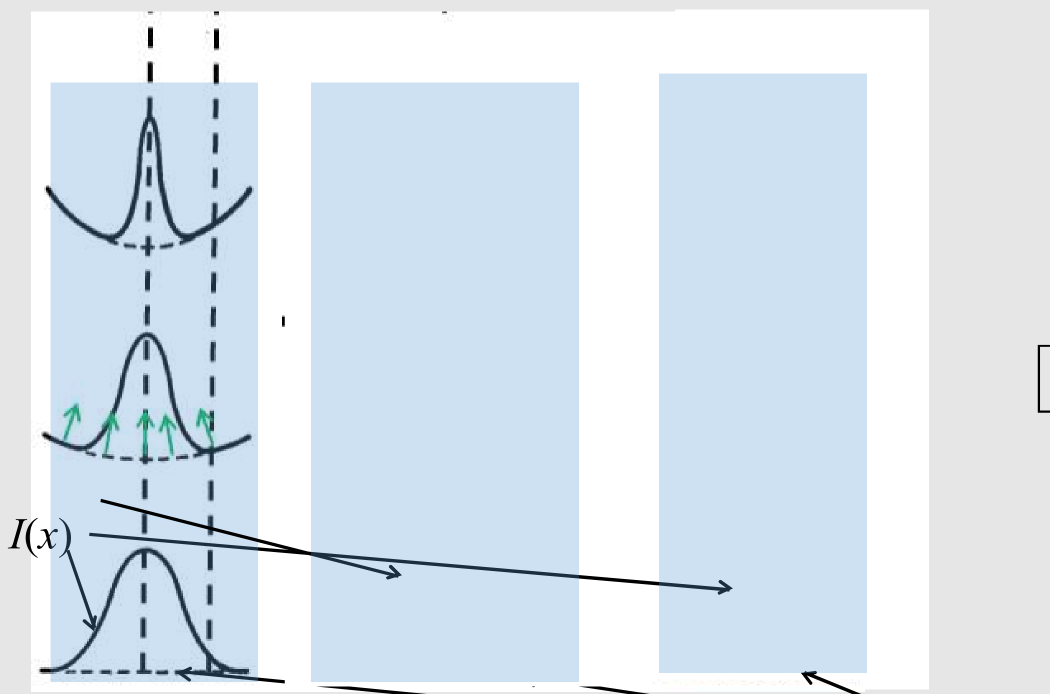
1. An *optical* soliton is a shape invariant self-trapped beam of light or a *self-induced waveguide*

2. Solitons occur frequently in nature in all nonlinear wave phenomena

1D Bright Spatial Soliton

Diffraction in 1D only!

Optical Kerr Effect → Self-Focusing: $n(\mathrm{I})=n_0+n_2\mathrm{I},\ n_2>0$



Self-focusing

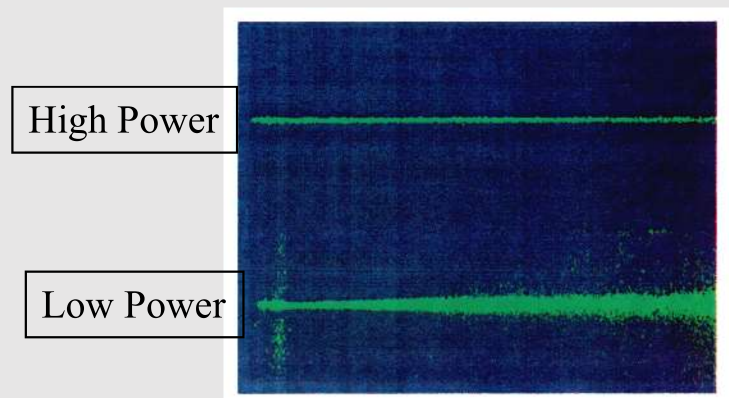
$n_2>0$

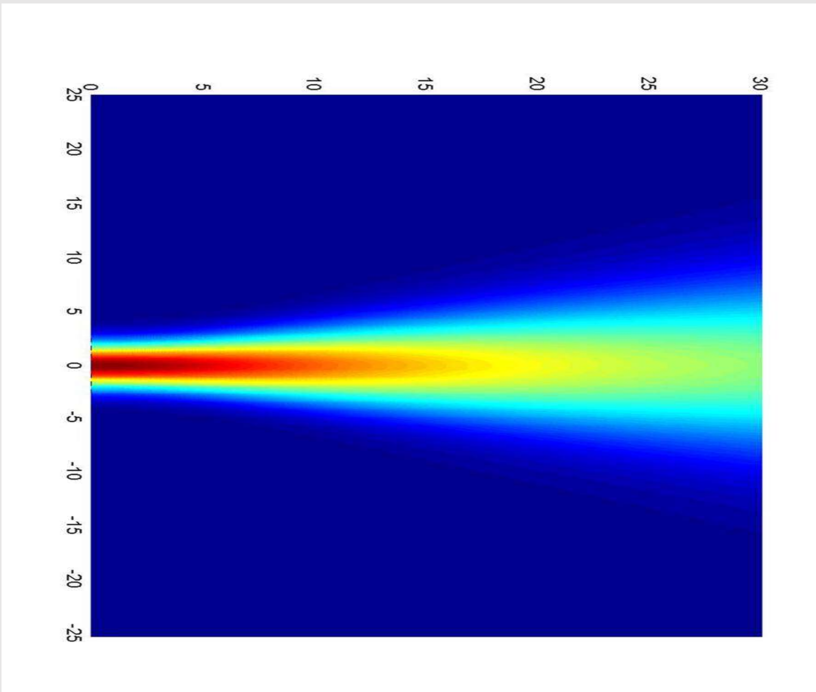
Diffraction in space

Soliton!

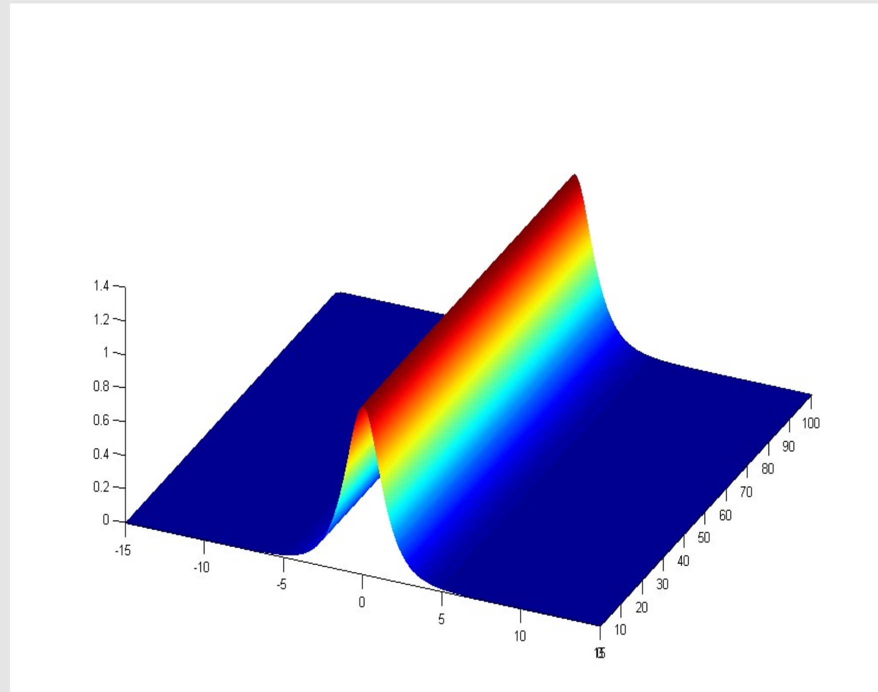
Soliton Properties

No change in shape on propagation





Berkas linear (*Linear beam*)



Soliton spasial

PERSAMAAN MAXWELL

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = J + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

E : medan elektrik
D : rapat fluks elektrik
H : medan magnet
B : rapat fluks magnetik
J : arus bebas (rapat arus)
ρ : muatan bebas (rapat muatan)

Untuk material non-magnetik:

$$\rho = 0; \quad \mathbf{J} = 0$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

P : vektor polarisasi dalam medium
ε_0 : permitivitas ruang hampa
μ_0 : permeabilitas ruang hampa

Material Kerr:

$$\mathbf{D} = \varepsilon_0 \left(1 + \chi^{(1)} + \chi^{(3)} \,|\, \mathbf{E} \,|^2 \right)\mathbf{E}$$

$$\nabla^2 \mathbf{E} - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\left(\left(1 + \chi^{(1)} + \chi^{(3)} \,|\, \mathbf{E} \,|^2 \right)\mathbf{E}\right) = \nabla\left(\nabla \cdot \mathbf{E}\right).$$

- TE Mode
- Arah perambatan sumbu-z:

$$\mathbf{E} = [0, E_y(x,z,t), 0]$$

$$\mathbf{H} = \left[H_x(x,z,t), 0, H_z(x,z,t)\right]$$

$$\nabla \bullet \mathbf{E} = 0$$

$$\left(\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial z^2}\right)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\left(1+\chi^{(1)}+\chi^{(3)}\,|E_y|^2\right)E_y\right)=0,$$

$$c=1/\sqrt{\varepsilon_0\mu_0}$$

Asumsi gel.monokromatik: $E_y(x,z,t)=E(x,z)\exp(-i\omega t)$

Pers. Helmholtz nonlinear:

$$\boxed{\left(\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial z^2}\right)+\frac{\omega^2}{c^2}n^2E=0}$$

$$n^2=\left(n_0+n_2\,|E|^2\right)^2\approx n_0^2+2n_0n_2\,|E|^2$$

$n_0=\sqrt{1+\chi^{(1)}}$: *indeks bias linear*

$\Delta n=n_2\,|E|^2$: *perub. indeks bias*

$n_2=\dfrac{\chi^{(3)}}{2n_0}$: *koef. nonlinear*

$$E(x,z) = B(x,z)\exp(ik_0 z) + cc$$

$$\boxed{\frac{i}{k_0}\frac{\partial B}{\partial z} + \frac{1}{2k_0^2}\frac{\partial^2 B}{\partial x^2} + \cancel{\frac{1}{2k_0^2}\frac{\partial^2 B}{\partial z^2}} + \frac{n_2}{n_0}|B|^2 B = 0,}$$

B(x,z) selubung gelombang dengan asumsi SVEA (*Slowly Varying Envelope Approximation)*

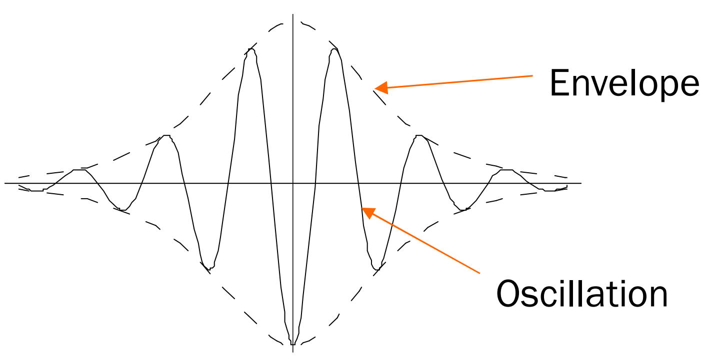
SVEA

Multiple-scaled

$$X = \delta k_0 x \ \ dan \ \ Z = \delta^2 k_0^2 z$$

$$B(x,z) = \delta\sqrt{n_0 / n_2}\, A(X,Z)$$

$$\delta = 1/k_0 w_0 \qquad (0<\delta<<1)$$



Pers. *Nonlinear Schrodinger/ NLS:*

$$i\frac{\partial A}{\partial Z}+\frac{1}{2}\frac{\partial^2 A}{\partial X^2}+sign(n_2)|A|^2 A=0$$

$$sign(n_2)=\begin{cases}+1; & n_2\geq 0\\ -1; & n_2<0\end{cases}.$$

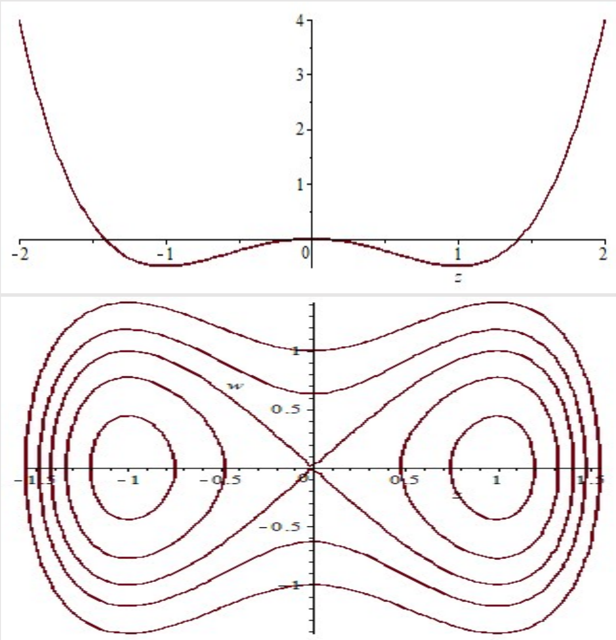
Diasumsikan bahwa selubung gelombang berbentuk

$$A(X,Z)=f(X)\exp(i\beta Z);\ \ f,\beta\in\Re$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{d^2 f}{dX^2}\right)-\beta f+sign(n_2)f^3=0$$

$$\left(\frac{df}{dX}\right)^2 - 2\beta f^2 + sign(n_2) f^4 = C$$

C = konstanta



Newton equation: $U = sign(n_2) f^4 - 2\beta f^2$

f : variabel posisi partikel

X : variabel waktu

- Solusi soliton terang (***Bright Soliton***)

$sign(n_2) = +1$

$$A(X, Z) = \text{sech}(X)\exp\left(\frac{1}{2} iZ\right).$$

- Solusi soliton gelap (***Dark Soliton***)

$sign(n_2) = -1$

$$A(X, Z) = \tanh(X)\exp(-iZ).$$

TRANSFORMASI PENYELESAIAN

$$A_1(X,Z) = \eta A(\eta X, \eta^2 Z)$$

$$A_1(X,Z) = \eta\,\mathrm{sech}(\eta X)\exp\left(\frac{1}{2} i \eta^2 Z\right)$$ Bright Soliton

$$A_1(X,Z) = \eta \tanh(\eta X)\exp(-i\eta^2 Z)$$ Dark Soliton

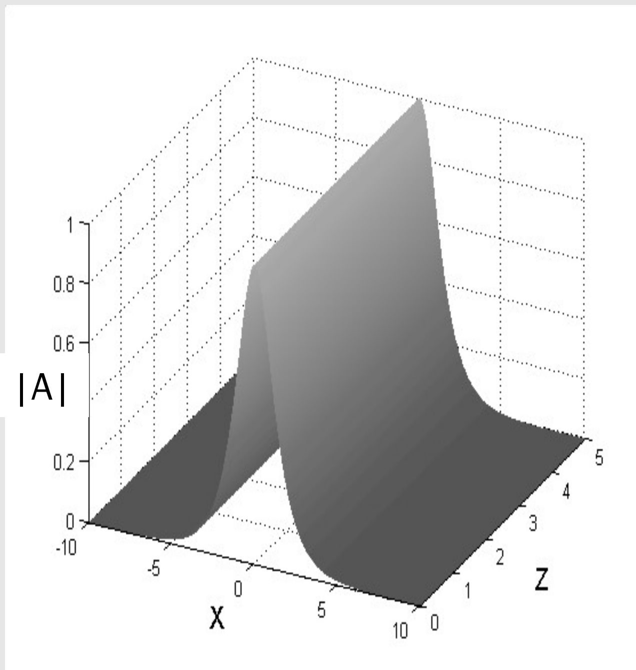
$$A_2(X,Z) = A(X - VZ, Z)\exp\left(iVX - i\frac{1}{2}V^2 Z\right)$$

Bright Soliton

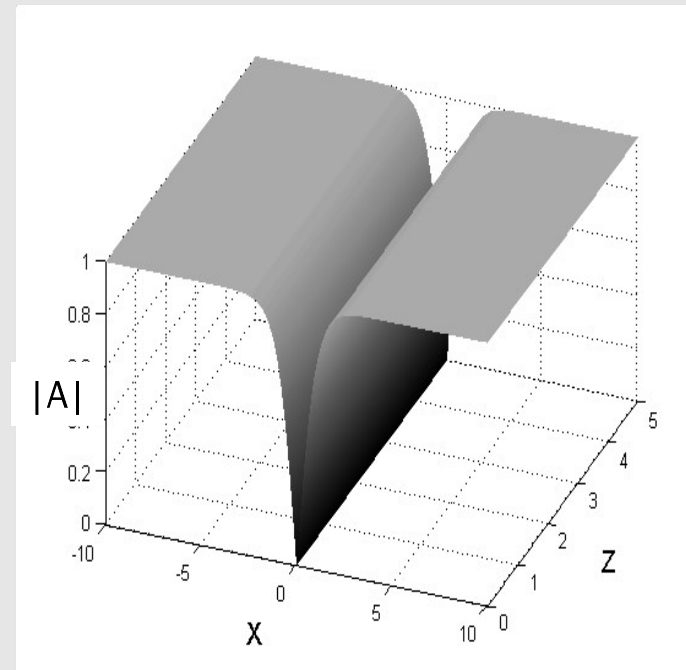
$$\begin{aligned} A_2(X,Z) &= \eta\,\mathrm{sech}(\eta(X - VZ))\exp\left(\frac{1}{2} i\eta^2 Z\right)\exp\left(iVX - \frac{1}{2} iV^2 Z\right) \\ &= \eta\,\mathrm{sech}(\eta(X - VZ))\exp\left(iVX + \frac{1}{2} i(\eta^2 - V^2)Z\right) \end{aligned}$$

Dark Soliton

$$\begin{aligned} A_2(X,Z) &= \eta \tanh(\eta(X - VZ))\exp(-i\eta^2 Z)\exp\left(iVX - \frac{1}{2} iV^2 Z\right) \\ &= \eta \tanh(\eta(X - VZ))\exp\left(iVX - \frac{1}{2} i(2\eta^2 + V^2)Z\right) \end{aligned}$$



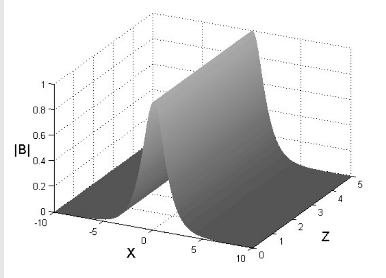
Gambar 1. Soliton terang stasioner.



Gambar 2. Soliton gelap stasioner.

Transformasi Solusi

$$A(X,Z) = \operatorname{sech}(X)\exp\left(\frac{1}{2}iZ\right).$$



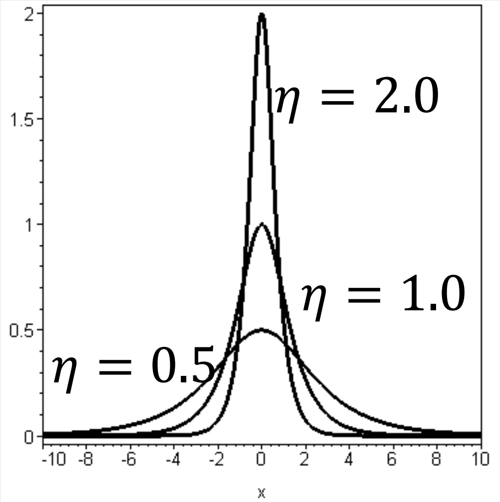
Jika $A_1(X,z)$ adalah solusi, maka fungsi-fungsi $A_2(X,Z)$ berikut juga solusi:

$$A_2(X,Z) = \eta A_1\left(\eta X, \eta^2 Z\right)$$

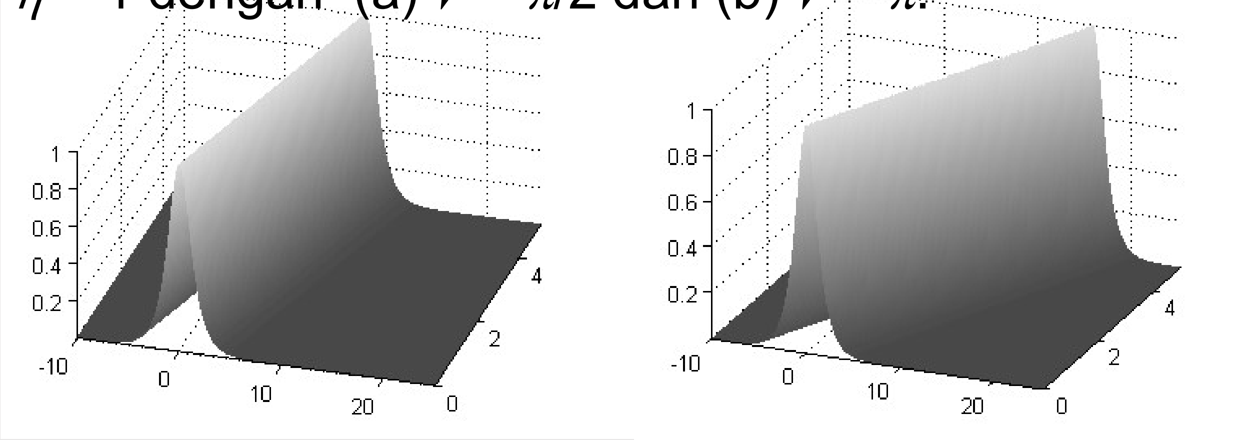
$$A_2(X,Z) = A_1(X - VZ, Z)\exp\left(iVX - i\frac{1}{2}V^2 Z\right)$$

$$A(X,Z) = \eta \operatorname{sech}(\eta X)\exp\left(\frac{1}{2}i\eta^2 Z\right)$$

$$A(X,Z) = \eta \operatorname{sech}(\eta(X - VZ))\exp\left(iVX + \frac{1}{2}i\left(\eta^2 - V^2\right)Z\right)$$

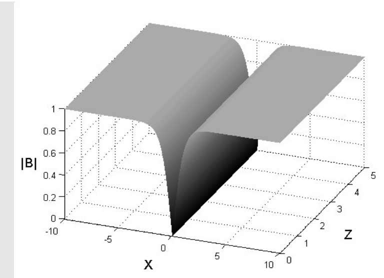


η = 1 dengan (a) $V = \pi/2$ dan (b) $V = \pi$.



Transformasi Solusi

$$A(X,Z) = \tanh(X)\exp(iZ).$$



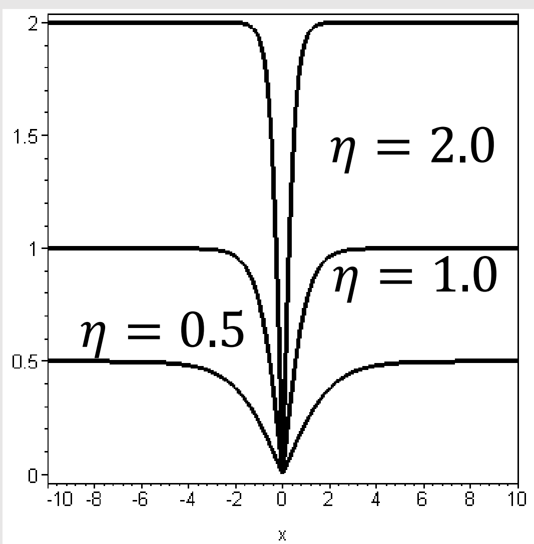
Jika $A_1(X,z)$ adalah solusi, maka fungsi-fungsi $A_2(X,Z)$ berikut juga solusi:

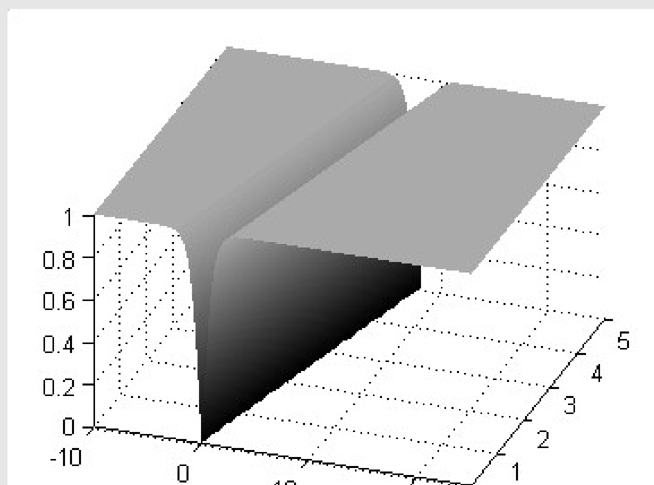
$$A_2(X,Z) = \eta A_1\left(\eta X, \eta^2 Z\right)$$

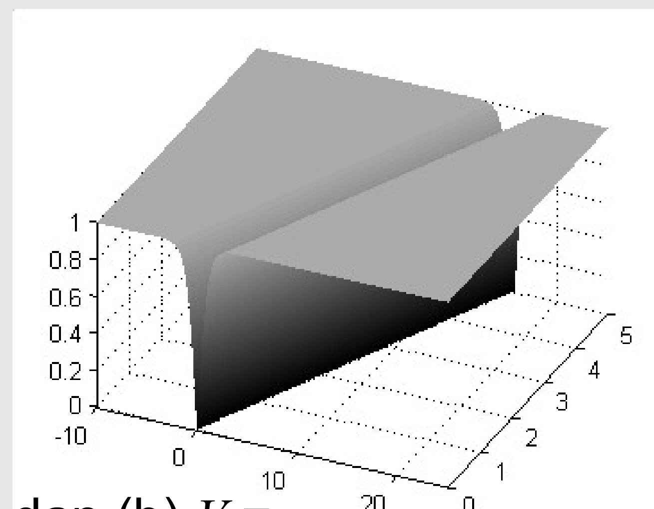
$$A_2(X,Z) = A_1(X - VZ, Z)\exp\left(iVX - i\frac{1}{2}V^2 Z\right)$$

$$A(X,Z) = \eta\tanh(\eta X)\exp\left(-i\eta^2 Z\right)$$

$$A(X,Z) = \eta\tanh(\eta(X - VZ))\exp\left(iVX - \frac{1}{2}i\left(2\eta^2 + V^2\right)Z\right)$$







η = 1 dengan (a) $V = \pi/2$ dan (b) $V = \pi$.

SKEMA CRANK-NICHOLSON UNTUK PERSAMAAN NLS

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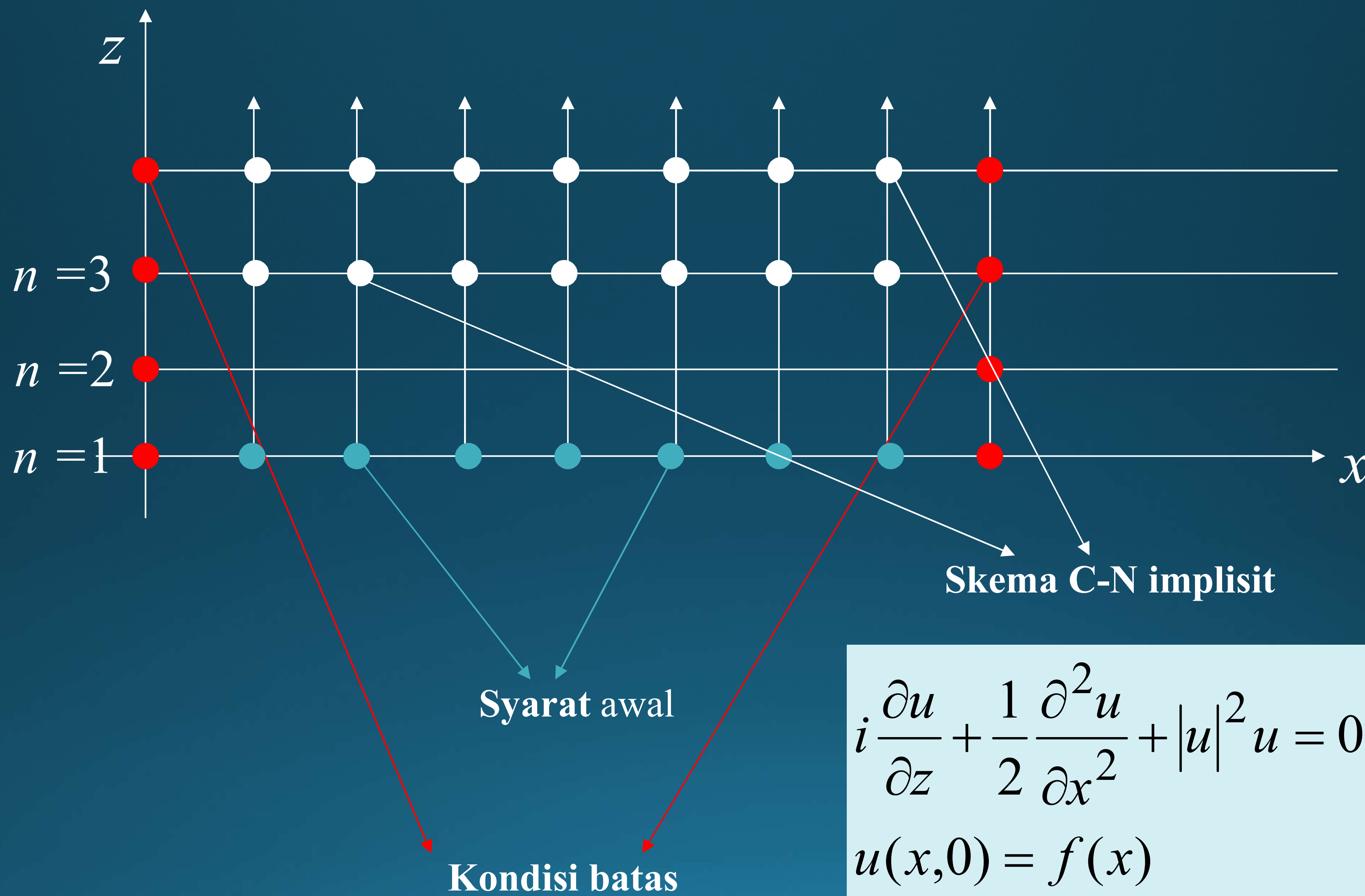
Skema Crank-Nicolson (C-N) *Implisit*



$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2 u=0$$

$$u(x,0)=f(x)$$

Skema Crank-Nicolson Implisit

Skema C-N:

$$\frac{\partial u(x,t)}{\partial t} = f(u(x,t)) \Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} = \tfrac{1}{2} f(u_j^{n+1}) + \tfrac{1}{2} f(u_j^n)$$

Contoh (persamaan panas 1D):

$$u_t = u_{xx} \Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2\Delta x^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x^2}$$

➔ Kesalahan pemotongan $T = O\left(\Delta x^2, \Delta t^2\right)$

Skema Crank-Nicolson Implisit

Persamaan NLS:

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2u=0$$

$$i\frac{u_j^{n+1}-u_j^n}{\Delta z}+\frac{1}{4}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}+\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\Delta x^2}\right)$$
$$+\frac{1}{4}\left(\left|u_j^{n+1}\right|^2+\left|u_j^n\right|^2\right)\left(u_j^{n+1}+u_j^n\right)=0.$$

Kondisi batas Dirichlet:

$$u(x=-L,z)=u(x=L,z)=0 \Longrightarrow u_1^n=u_{M+1}^n=0$$

$$\boxed{P\left(\vec{U}^{\,n+1}\right)\vec{U}^{\,n+1}=Q\left(\vec{U}^{\,n+1}\right)\vec{U}^{\,n}}$$

$$P\left(U^{n+1}\right)=\begin{bmatrix} p\left(u_2^{n+1}\right) & \frac{1}{4}\gamma & 0 & \cdots & 0 \\ \frac{1}{4}\gamma & p\left(u_3^{n+1}\right) & \frac{1}{4}\gamma & \cdots & 0 \\ 0 & \frac{1}{4}\gamma & p\left(u_4^{n+1}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p\left(u_M^{n+1}\right) \end{bmatrix}$$

$$U^n=\left(u_2^n,u_3^n,\ldots,u_M^n\right)^T$$

$$p\left(u_j^{n+1}\right)=i-\frac{1}{2}\gamma+\frac{1}{4}\Delta z\left(\left|u_j^{n+1}\right|^2+\left|u_j^n\right|^2\right)$$

$$Q\left(U^{n+1}\right)=\begin{bmatrix} q\left(u_2^{n+1}\right) & -\frac{1}{4}\gamma & 0 & \cdots & 0 \\ -\frac{1}{4}\gamma & q\left(u_3^{n+1}\right) & -\frac{1}{4}\gamma & \cdots & 0 \\ 0 & -\frac{1}{4}\gamma & q\left(u_4^{n+1}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q\left(u_M^{n+1}\right) \end{bmatrix}$$

$$q\left(u_j^{n+1}\right)=i+\frac{1}{2}\gamma-\frac{1}{4}\Delta z\left(\left|u_j^{n+1}\right|^2+\left|u_j^n\right|^2\right)$$

$$\gamma=\Delta z/\Delta x^2.$$

$$P\left(\vec{U}^{n+1}\right)\vec{U}^{n+1} = Q\left(\vec{U}^{n+1}\right)\vec{U}^{n}$$

➔ Sistem persamaan Nonlinear

Metode Iterasi ➔

$$P\left(\left(\vec{U}^{n+1}\right)^{s}\right)\left(\vec{U}^{n+1}\right)^{s+1} = Q\left(\left(\vec{U}^{n+1}\right)^{s}\right)\vec{U}^{n}$$

Iterasi awal ➔

$$\left(\vec{U}^{n+1}\right)^{0} = \vec{U}^{n}$$

Kriteria penghentian ➔

$$\max_{j}\left|\left(\vec{U}_{j}^{n+1}\right)^{s+1} - \left(\vec{U}_{j}^{n+1}\right)^{s}\right| < 10^{-6}$$

Penyelesaian numerik ➔

$$\vec{U}^{n+1} = \left(\vec{U}^{n+1}\right)^{s+1}$$

$$\frac{i}{\Delta Z}\left(\vec{U}_0^{n+1}-\vec{U}_0^{n}\right)+\frac{1}{4\Delta X^2}\left(\vec{U}_+^{n+1}-2\vec{U}_0^{n+1}+\vec{U}_-^{n+1}+\vec{U}_+^{n}-2\vec{U}_0^{n}+\vec{U}_-^{n}\right)$$

$$+\vec{\alpha}^{n}\circ\left(\vec{U}_0^{n+1}+\vec{U}_0^{n}\right)=\vec{\mathbf{0}}$$

$$\vec{U}_-^{n}=\left(u_1^n,u_2^n,\ldots,u_{M-1}^n\right)^T$$

$$\vec{U}_0^{n}=\left(u_2^n,u_3^n,\ldots,u_M^n\right)^T$$

$$\vec{U}_+^{n}=\left(u_3^n,u_4^n,\ldots,u_{M+1}^n\right)^T$$

$$\vec{\alpha}^{n}=\left(\alpha_2^n,\alpha_3^n,\ldots,\alpha_M^n\right)^T$$

$$\vec{\mathbf{0}}=\left(0,0,\ldots,0\right)^T$$

$$\alpha_j^n=\frac{1}{4}\left(\left|u_j^{n+1}\right|^2+\left|u_j^n\right|^2\right),\ j=2,3,\ldots,M$$

$$\boxed{\vec{u} \bullet \vec{v} = \left(u_1 v_1, u_2 v_2, \ldots, u_{M-1} v_{M-1}\right)^T}$$

$$\vec{u} = \left(u_1, u_2, \ldots, u_{M-1}\right)^T \qquad \vec{v} = \left(v_1, v_2, \ldots, v_{M-1}\right)^T$$

$$\boxed{\left\langle \vec{u}, \vec{v} \right\rangle = \vec{u} \bullet \vec{v}^{\,*} = \sum_{j=1}^{M-1} u_j v_j^*}$$

$$u_j^{n+1/2} = \tfrac{1}{2}\left(u_j^{n+1} + u_j^n\right)$$

$$\boxed{\begin{aligned} &\frac{i}{\Delta z}\left\langle \vec{U}_0^{n+1} - \vec{U}_0^n, 2\vec{U}_0^{n+\frac{1}{2}} \right\rangle + \frac{1}{4\Delta z^2}\left\langle \vec{U}_+^{n+1} - 2\vec{U}_0^{n+1} + \vec{U}_-^{n+1} + \vec{U}_+^n - 2\vec{U}_0^n + \vec{U}_-^n, 2\vec{U}_0^{n+\frac{1}{2}} \right\rangle \\ &\quad + \left\langle \vec{\alpha}^n \circ \left(\vec{U}_0^{n+1} + \vec{U}_0^n\right), 2\vec{U}_0^{n+\frac{1}{2}} \right\rangle = \vec{\mathbf{0}}. \end{aligned}}$$

$$\frac{i}{2\Delta z}\left(\sum_{j=1}^{M}\left|u_j^{n+1}\right|^2-\sum_{j=1}^{M}\left|u_j^{n}\right|^2\right)+\frac{i}{2\Delta z}\left(\sum_{j=1}^{M}u_j^{n+1}\left(u_j^{n}\right)^*-u_j^{n}\left(u_j^{n+1}\right)^*\right)$$

$$-\frac{1}{\Delta x^2}\sum_{j=1}^{M}\left|u_{j+1}^{n+\frac{1}{2}}-u_j^{n+\frac{1}{2}}\right|^2+\sum_{j=1}^{M}\alpha_j^n\left|u_j^{n+\frac{1}{2}}\right|^2=0$$

$$\sum_{j=1}^{M}\left|u_j^{n+1}\right|^2=\sum_{j=1}^{M}\left|u_j^{n}\right|^2,\forall n.$$

$$P_{NLS}=\int_{-\infty}^{\infty}\left|u\right|^2\mathrm{dx}$$

$$\equiv \text{konstan}$$

$$P_h=\sum_{j=1}^{M}\left|u_j^{n}\right|^2=\text{konstan; untuk semua } n$$

skema Crank-Nicolson implisit tetap menjaga hukum konservasi