# DISTANCE IRREGULAR VERTEX LABELING OF GRAPH

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**Universitas Jember, Indonesia**

**Workshop KK Kombinatorika: Pewarnaan dan Pelabelan pada Graf**
**Jurusan Matematika Universitas Andalas**
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# Outline

A Introduction

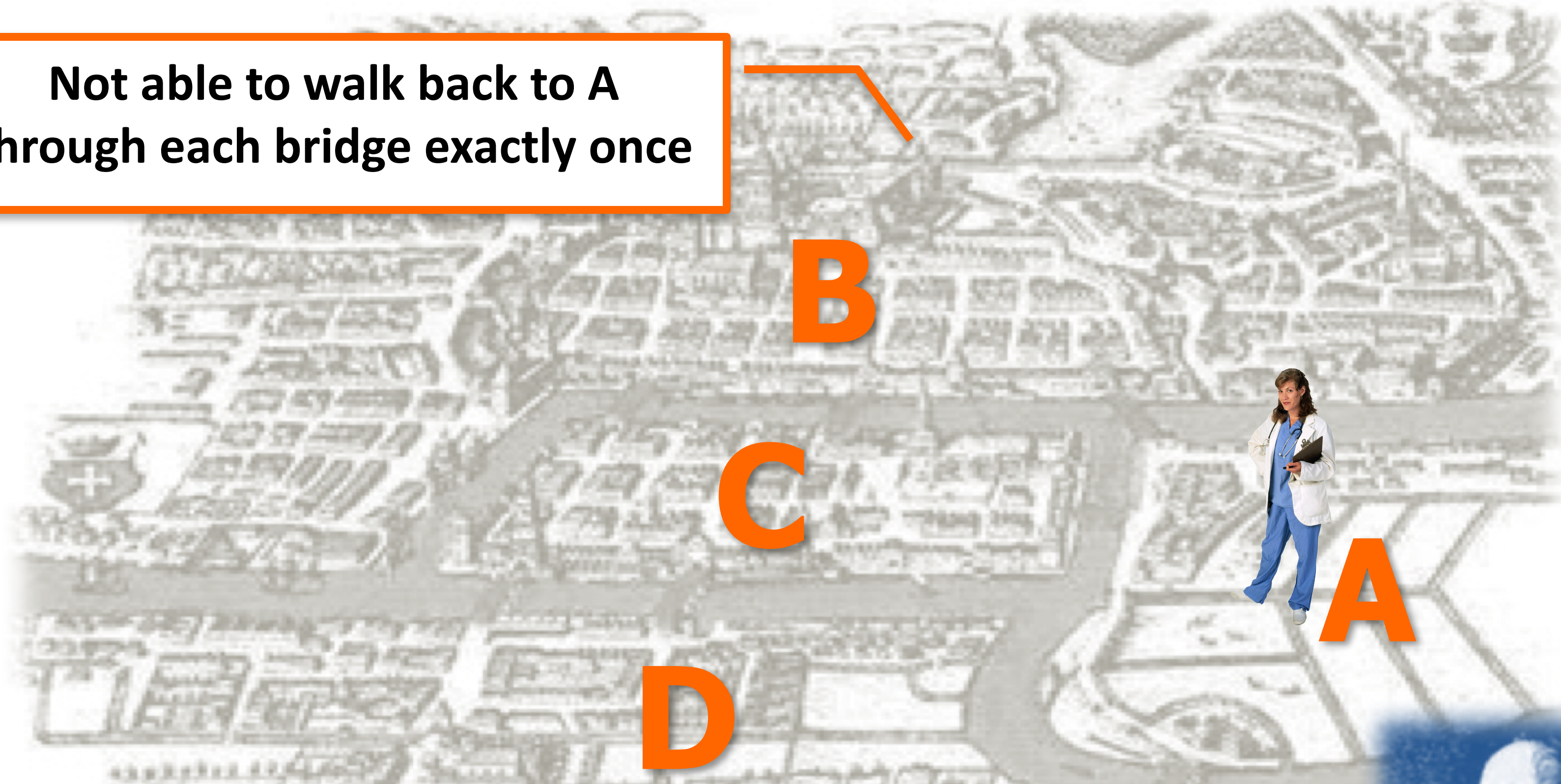
B Distance irregular labeling

C Inclusive distance irregular labeling

D Open Problems

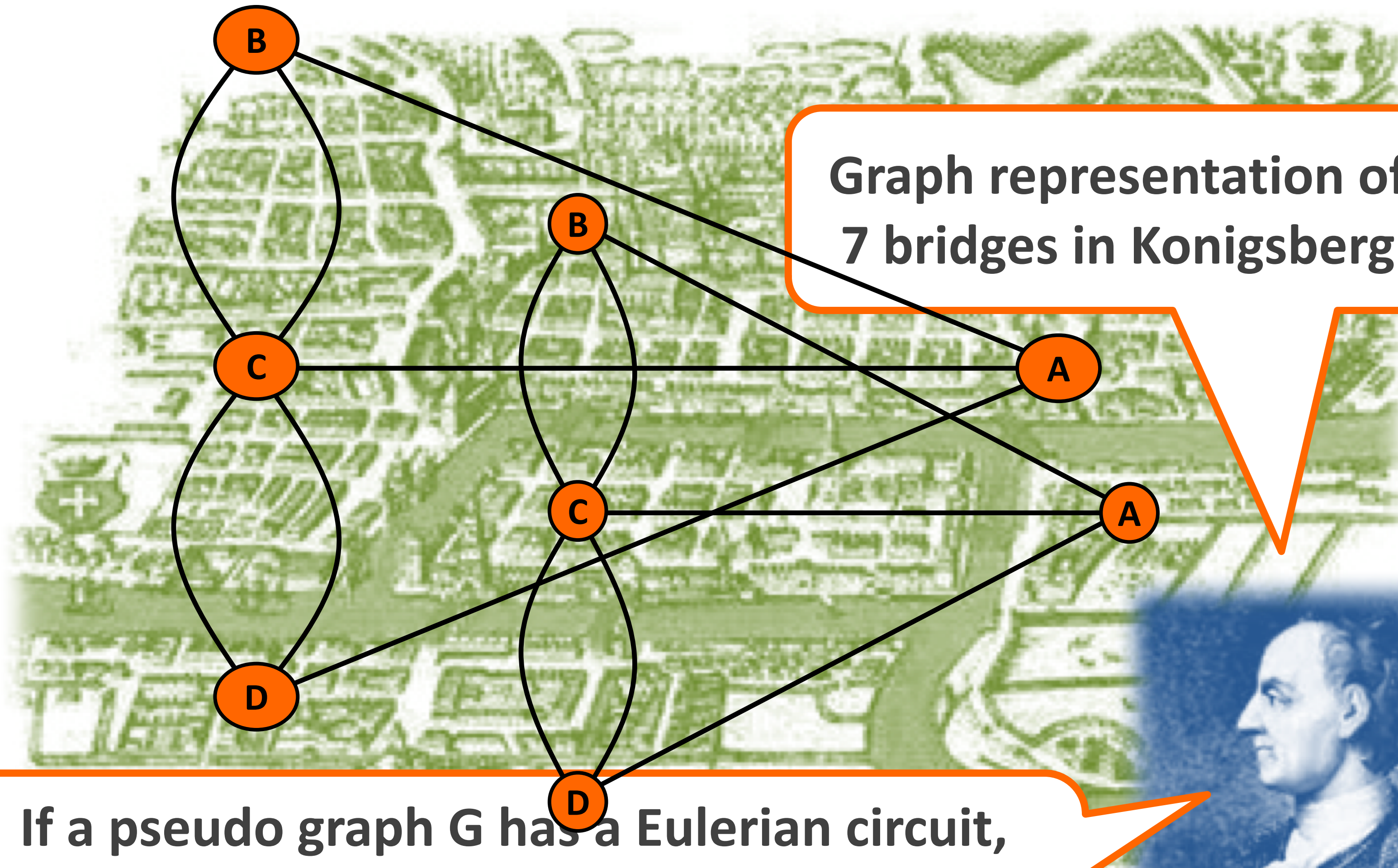
# History of Graph Theory

Not able to walk back to A through each bridge exactly once




Impossible because there are regions connection by odd numbers of bridge

# INTRODUCTION History of Graph Theory




Graph representation of 7 bridges in Konigsberg

If a pseudo graph G has a Eulerian circuit, then the graph is connected and the degree of every vertex is even

## Development of Graph Theory

Since then many other concepts in Graph Theory have sprung up to solve problems in daily life.

## Graph Labelings

Graph labeling is growing very fast with several type and broad range of applications

## Irregular assignment of graph

- G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba (1988)
- M. Bačá, S. Jendrol', M. Miller, and J. Ryan (2007)

## Distance magic labeling of graphs

- M. Miller, C. Rodger, R. Simanjuntak (2003)

## (a,d)-distance antimagic graphs

- S. Arumugam, N. Kamatchi (2012)

# Distance Irregular Vertex Labeling

## Definition

A *distance irregular vertex labeling* of the graph $G$ with $v$ vertices is an assignment $\lambda : V \rightarrow \{1, 2, \ldots, k\}$ so that the weights calculated at vertices are distinct.

## Weight

The *weight* of a vertex $x$ in $G$ is defined as the sum of the labels of all the vertices adjacent to $x$ (distance 1 from $x$), that is,

$$wt(x) = \sum_{y \in N(x)} \lambda(y)$$

## dis(*G*)

The *distance irregularity strength* of $G$, denoted by $dis(G)$, is the minimum value of the largest label $k$ over all such irregular assignments.

**(Slamin, 2017)**

# Distance Irregular Vertex Labeling

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## On distance-irregular labelings of cycles and wheels

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*In memory of Mirka Miller*

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# On inclusive distance vertex irregular labelings

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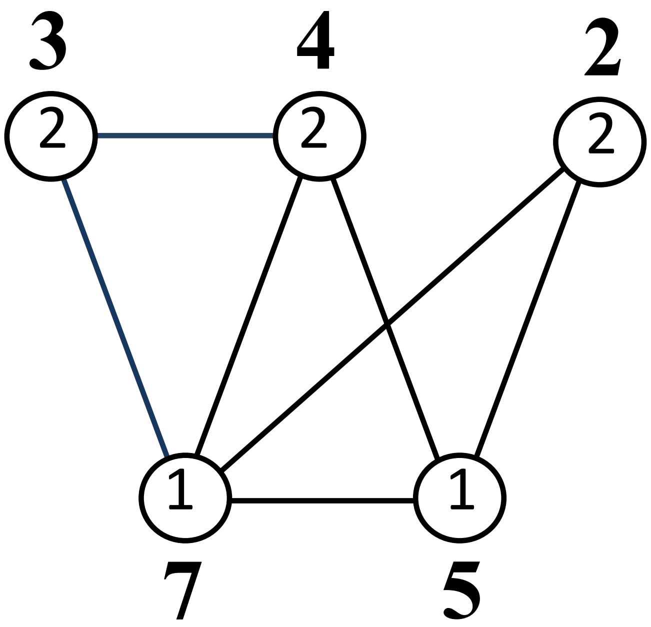
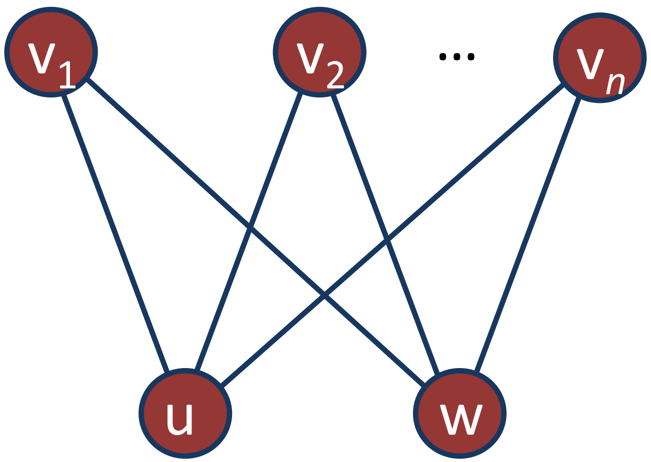
## Example

A distance irregular vertex labeling of the graph *G* with 5 vertices with dis(*G*) = 2.



## Observation 1

Let *u* and *w* be any two distinct vertices in a connected graph *G*. If *u* and *w* have identical neighbors, i.e., $N(u) = N(w)$, then *G* has no distance irregular vertex labeling.



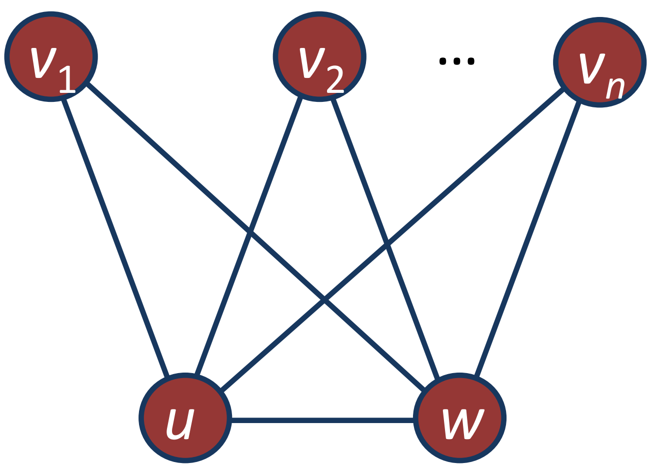
**Graphs that have no distance irregular labeling:**

- Complete bipartite graphs $K_{m,n}$ for any $m,n \geq 2$.
- Complete multipartite graphs $H_{m,n}$ for any $m,n \geq 2$.
- Stars on $n$+1 vertices $S_n$ for $n \geq 2$.
- Trees $T_n$ for any n ≥ 3, that contain a vertex with at least two leaves.

# Other Necessary Condition

## Observation 2

Let $u$ and $w$ be two adjacent vertices in a connected graph $G$. If $N(u) - \{w\} = N(w) - \{u\}$, then the labels of $u$ and $w$ must be distinct, that is, $\lambda(u) \neq \lambda(w)$.



## Lemma

Let $G$ be a connected graph on $v$ vertices with minimum degree $\delta$ and maximum degree $\Delta$ and there is no vertex having identical neighbors. Then

$$dis(G) \geq \left\lceil \frac{v+\delta-1}{\Delta} \right\rceil.$$

## Proof

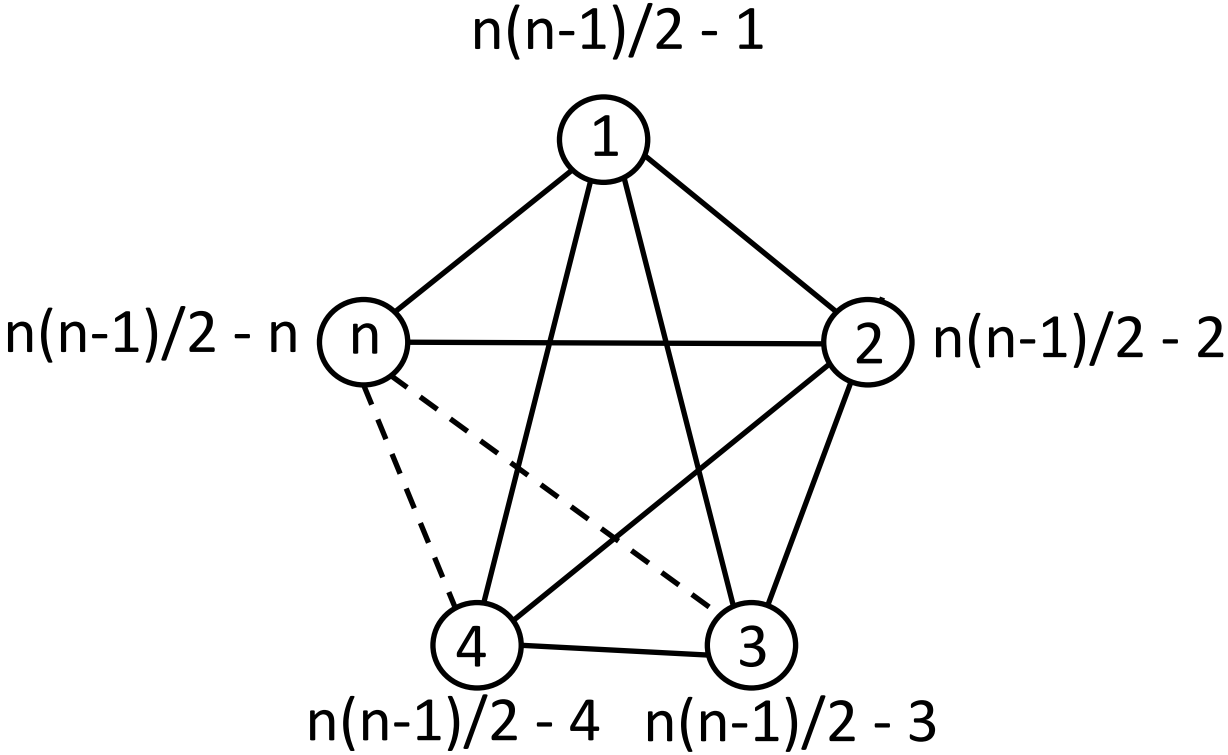
- The smallest weight is $\delta$.
- Since the weight must be distinct, the largest weight is at least $v + \delta - 1$.
- This weight is obtained from the sum of at most $\Delta$ integers.
- Thus the largest label that contributes to this weight must be at least $\left\lceil \frac{v+\delta-1}{\Delta} \right\rceil$.

**(Slamin, 2017)**

# Complete Graphs

## Theorem

Let $K_n$ be a complete graph with $n \geq 3$ vertices. Then $\mathrm{dis}(K_n) = n$.




**(Slamin, 2017)**

# Paths

**Theorem**

Let $P_n$ be a path with $n \geq 4$ vertices. Then

$$dis(P_n) = \left\lceil \frac{n}{2} \right\rceil$$

**(Slamin, 2017)**

## Theorem

Let $C_n$ be a cycle with $n \geq 5$ vertices. Then

$$dis(C_n) = \left\lceil \frac{n+1}{2} \right\rceil \text{ if } n = 0,1,2,5 \mod 8.$$

**(Slamin, 2017)**

$$dis(C_n) = \begin{cases} \frac{n+3}{2} & if\, n \equiv 3,7 \mod 8 \\ \lceil \frac{n+1}{2} \rceil & if\, n \equiv 4,6 \mod 8 \end{cases}$$

**(N.H. Bong, Y. Lin, Slamin, 2017+)**

**Theorem**

Let $W_n$ be a wheel with $n \geq 5$ rim vertices. Then

$$dis(W_n) = \left\lceil \frac{n+1}{2} \right\rceil \quad \text{if } n = 0,1,2,5 \mod 8.$$

**(Slamin, 2017)**

$$dis(W_n) = \begin{cases} \frac{n+3}{2} & if\ n \equiv 3,7 \mod 8 \\ \lceil \frac{n+1}{2} \rceil & if\ n \equiv 4,6 \mod 8 \end{cases}$$

**(N.H. Bong, Y. Lin, Slamin, 2017+)**

## Theorem

Let $f_n$ be a friendship graph with $n \geq 3$ triangles. Then dis($f_n$)=2$n$.

## Proof

- Every vertex on the rim must have different label, otherwise there will be the same weight among the vertices
- Thus $dis(f_n) \geq 2n$.
- Label the 2$n$ vertices of $f_n$ as follows:

$$\lambda_1(c) = 1$$

$$\lambda_1(x_i) = 2i; \quad i = 1 \leq i \leq n$$

$$\lambda_1(y_i) = 2i - 1; \quad i = 1 \leq i \leq n$$

This gives the weights for vertices:

$$\omega_1(c) = \frac{2n(2n+1)}{2}$$

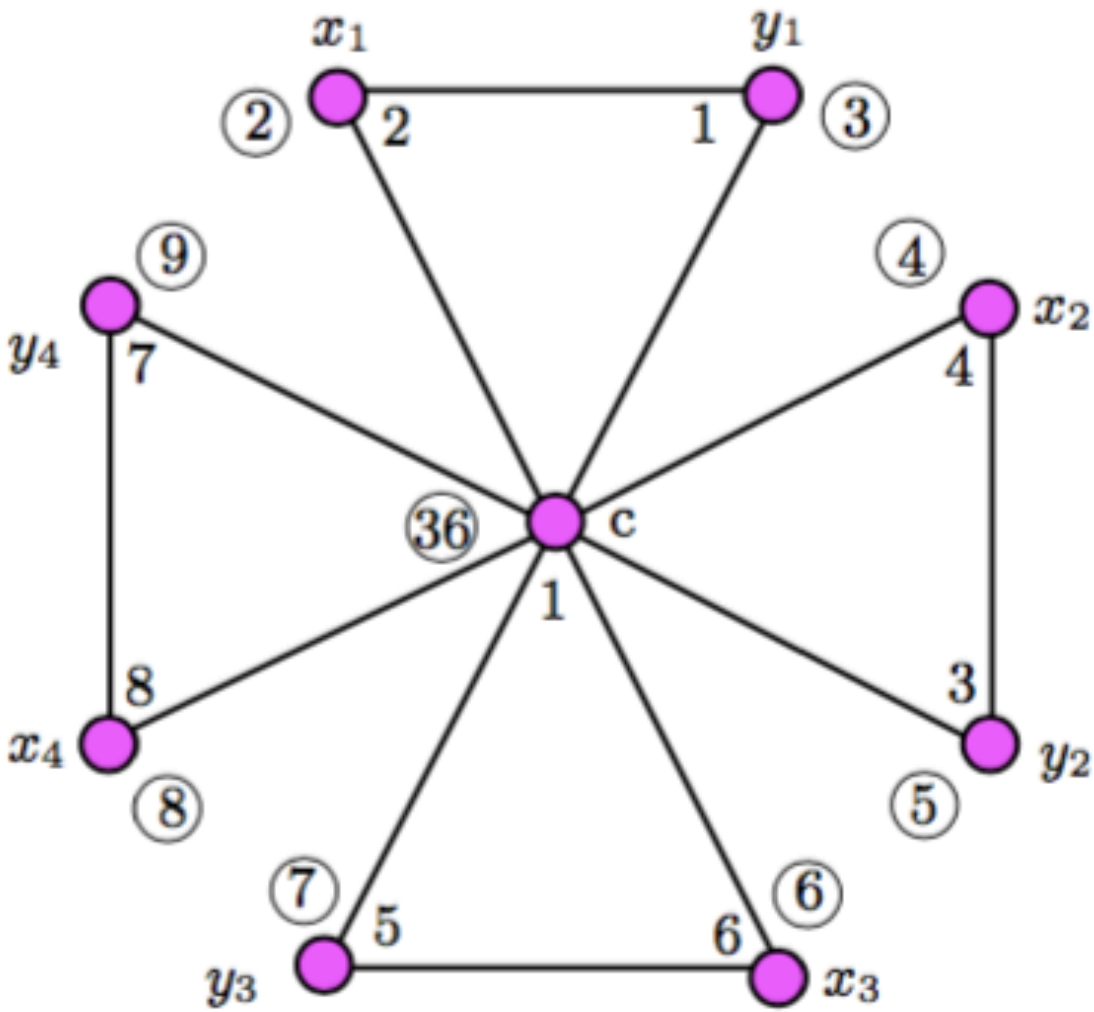
$$\omega_1(x_i) = 2i; \quad i = 1 \leq i \leq n$$

$$\omega_1(y_i) = 2i + 1; \quad i = 1 \leq i \leq n$$

- The labeling implies that $dis(f_n) \leq 2n$
- Therefore $dis(f_n) = 2n$

**(Slamin, T. Windartini, K.D. Purnomo, 2017+)**

# Friendship Graphs




$dis(f_4) = 8$

# Other Wheel Related Graphs

**Helm**

Let $H_n$ be a helm with $n \geq 3$ rim vertices. Then dis($H_n$) = $n$

**Flower**

Let $Fl_n$ be a flower with $n \geq 3$ rim vertices. Then dis($Fl_n$) = $n$

Fan

Let $F_n$ be a fan with $n \geq 4$ rim vertices. Then dis($F_n$) = $\lceil n/2 \rceil$

**(Slamin, T. Windartini, K.D. Purnomo, 2017+)**

## Definition

An *inclusive distance irregular vertex labeling* of the graph *G* with *v* vertices is an assignment $f : V \rightarrow \{1, 2, \ldots, k\}$ so that the weights calculated at vertices are distinct.

## Weight

The *weight* of a vertex *x* in *G* is defined as the sum of the label of *x* and the labels of all the vertices adjacent to *x* (distance 1 from *x*), that is,

$$wt_f(v) = f(v) + \sum_{u:1 \leq d(u,v) \leq d} f(u).$$

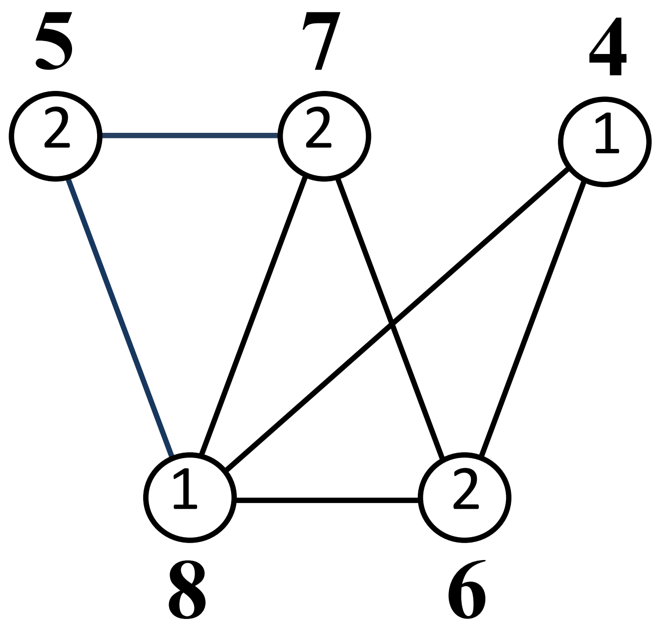
## $\overline{\text{dis}}(G)$

The *inclusive distance irregularity strength* of *G*, denoted by $\overline{\text{dis}}(G)$, is the minimum value of the largest label *k* over all such irregular assignments.

**(N.H. Bong, Y. Lin, Slamin, 2017+)**

## Example

Inclusive distance irregular vertex labeling of the graph $G$ with 5 vertices with $\overline{\text{dis}}(G) = 2$.



# Double Stars

**Theorem**

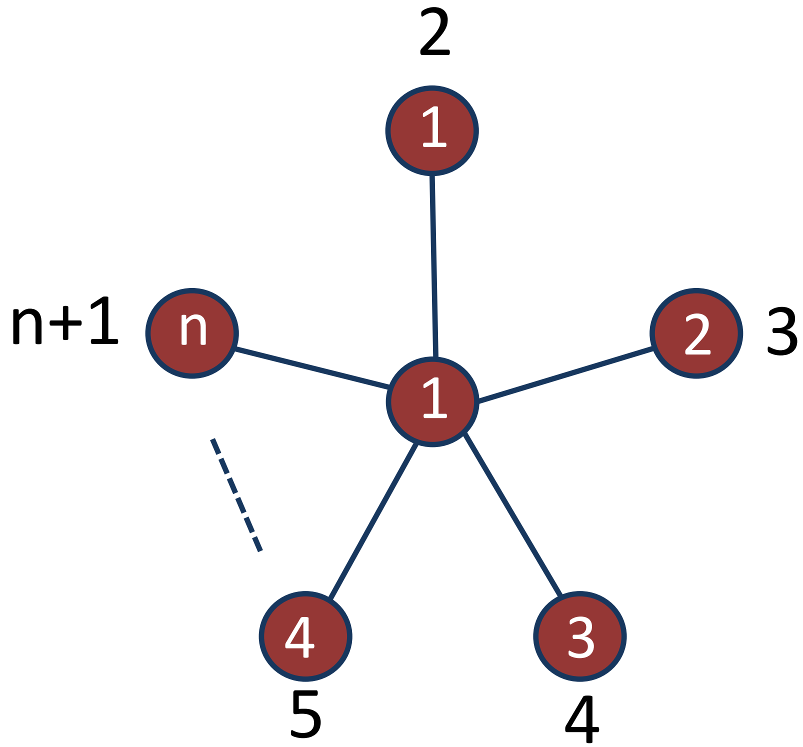
Let *S(m,n)* be a double star on *m* + *n* + 2 vertices. Then

$$\overline{\text{dis}}(S(m,n)) = \begin{cases} n & \text{if } m < n \\ n+1 & \text{if } m = n. \end{cases}$$

**(N.H. Bong, Y. Lin, Slamin, 2017+)**

**Theorem**

Let $S_n$ be a star on $n + 1$ vertices and n > 1. Then $\overline{\text{dis}}(S_n) = n$




**(N.H. Bong, Y. Lin, Slamin, 2017+)**

## Theorem

Let $P_n$ be a path on $n$ vertices. Then

$$\overline{\text{dis}}(P_n) = \begin{cases} \infty & \textit{for } n = 2, \\ 3 & \textit{for } n = 5, \\ \lceil \frac{n+1}{3} \rceil & \textit{for } n \not\equiv 2 \pmod 9, n \neq 5 \end{cases}$$

*and*

$$\tfrac{n+1}{3} \leq \overline{\text{dis}}(P_n) \leq \tfrac{n+1}{3} + 1,$$

*when* $n \equiv 2 \pmod 9$, $n \geq 11$.

**(M. Baca, A. Semanicova ́-Fenovcıkova, Slamin, K. A. Sugeng, 2017+)**

**Theorem**

Let $C_n$ be a cycle on $n$ vertices. Then

$$\overline{\text{dis}}(C_n) = \begin{cases} \infty & \textit{for } n = 3, \\ 4 & \textit{for } n = 4, \\ \lceil \frac{n+2}{3} \rceil & \textit{for } n \not\equiv 2, 3, 4 \pmod{18},\ n \geq 5 \end{cases}$$

*and*

$$\left\lceil \frac{n+2}{3} \right\rceil \leq \overline{\text{dis}}(C_n) \leq \left\lceil \frac{n+2}{3} \right\rceil + 1,$$

*when* $n \equiv 2, 3, 4 \pmod{18}$, $n \geq 20$.

**(M. Baca, A. Semanicová -Fenovcıkova, Slamin, K. A. Sugeng, 2017+)**

# Wheels

## Theorem

Let $W_n$ be a wheel on $n$ rim vertices. Then

$$\overline{\text{dis}}(W_n) = \begin{cases} \infty & \textit{for } n = 3, \\ 4 & \textit{for } n = 4, \\ \lceil \frac{n+2}{3} \rceil & \textit{for } n \not\equiv 2, 3, 4 \pmod{18},\ n \geq 5 \end{cases}$$

*and*

$$\lceil \tfrac{n+2}{3} \rceil \leq \overline{\text{dis}}(W_n) \leq \lceil \tfrac{n+2}{3} \rceil + 1,$$

*when* $n \equiv 2, 3, 4 \pmod{18}$, $n \geq 20$.

**(M. Baca, A. Semanicová -Fenovcıkova, Slamin, K. A. Sugeng, 2017+)**

# Open Problems

## Open Problem 1

Determine the (inclusive) distance irregularity strengths of some particular families of graphs.

## Open Problem 2

Expand the (inclusive) distance irregular labeling of graphs to the distance at least 2.

## Open Problem 3

Characterize the relationship between inclusive distance irregular labeling and (a,d)-distance antimagic labeling.

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# THANK YOU