Numerical techniques for tunami simulation

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Abstract

Partial differential equations derived from momentum equations and a continuity equation are spatially discretized. ODE solvers are applied to a resultant system of ordinary differential equations.

1 Reduction of governing equations

1.1 Shallow water equations

A tsunami simulation involves physical quantities such as the velocity of water (u, v, w) and the pressure p. Those quantities are functions of the coordinates (x, y, z) and the time t, and the conservation law leads to partial differential equation. Consider the Euler equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,, \tag{3}$$

and the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,. \tag{4}$$

Suppose that the mean sea level is defined by $z = 0$, that the sea surface is defined by $z = \zeta\,(x, y, t)$, and that the sea floor is defined by $z = h\,(x, y)$ (Figure 1). Let

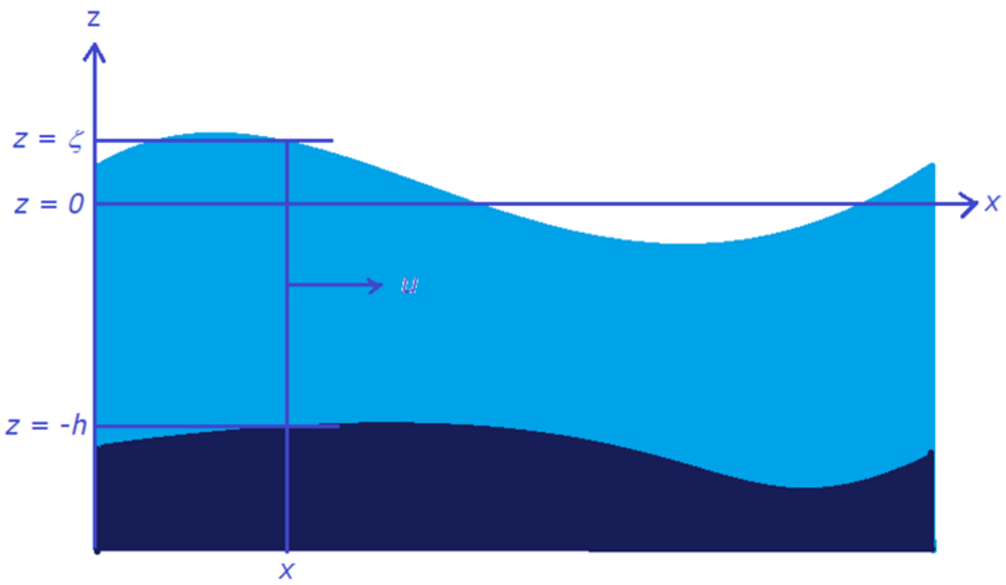

Figure 1: The mean sea level ($z = 0$), the sea surface ($z = \zeta(x, y, t)$), and the sea floor ($z = h(x, y)$).

$$M = \int_{-h(x,y)}^{\zeta(x,y,t)} u(x, y, z, t)\, dz\,, \quad N = \int_{-h(x,y)}^{\zeta(x,y,t)} v(x, y, z, t)\, dz\,. \tag{5}$$

The Euler equations (1) - (3), and the continuity equation (4) lead to the following system.

$$\frac{\partial M}{\partial t} + (h + \zeta)\frac{\partial \zeta}{\partial x} = 0\,, \tag{6}$$

$$\frac{\partial N}{\partial t} + (h + \zeta)\frac{\partial \zeta}{\partial y} = 0\,, \tag{7}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0\,, \tag{8}$$

1.2 Reduction over a triangular mesh

Consider a triangular mesh over a region R with n nodes (x_i, y_i) $(i = 1, 2, \ldots, n)$, and and m elements $(e_1, e_2, \ldots, e_n)$. Suppose $\Phi_i(x, y)$ is a basis function associated with the node (x_i, y_i). That is, $\Phi_i(x, y)$ is continuous

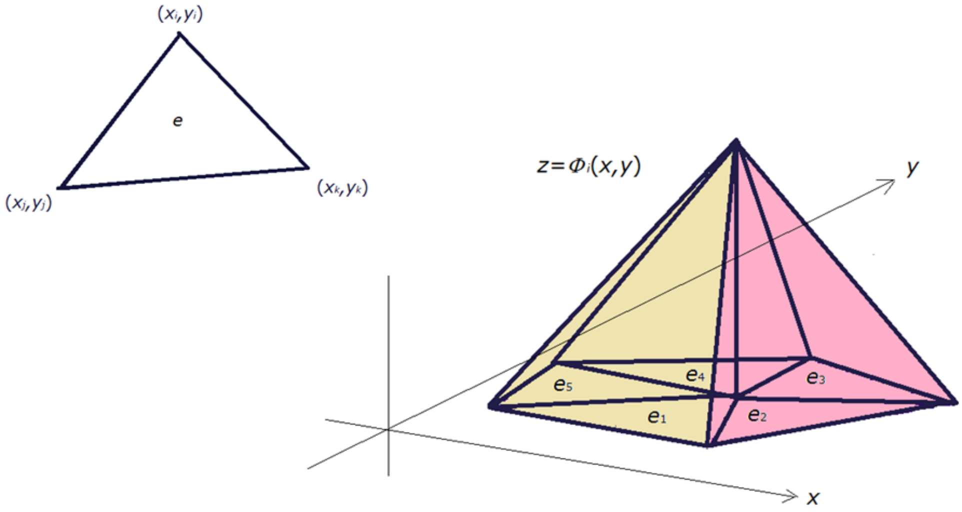

Figure 2: Basis function $\Phi_i\,(x, y)$.

over R and linear over each element, and satisfies the condition

$$\Phi_i\,(x_j, y_j) = \begin{cases} 1\,, & i = j\,, \\ 0\,, & i \neq j\,. \end{cases}$$

(Figure 2).

Suppose that $M\,(x, y, t)$, $N\,(x, y, t)$, $\zeta\,(x, y, t)$, and $h\,(x, y)$ are expressed in the following forms.

$$
\begin{aligned}
M(x,y,t) &= \sum_{j=1}^{n} M_j(t)\,\Phi_j(x,y)\,, && (9)\\
N(x,y,t) &= \sum_{j=1}^{n} N_j(t)\,\Phi_j(x,y)\,, && (10)\\
\zeta(x,y,t) &= \sum_{j=1}^{n} \zeta_j(t)\,\Phi_j(x,y)\,, && (11)\\
h(x,y) &= \sum_{j=1}^{n} h_j\Phi_j(x,y)\,. && (12)
\end{aligned}
$$

Substitution of those expressions into the equations (6), (7), and (8) leads to

$$
\begin{aligned}
\sum_{j=1}^{n}\frac{dM_j}{dt}\Phi_j(x,y) + g\frac{\partial\zeta}{\partial x}\sum_{j=1}^{n}(h_j+\zeta_j)\,\Phi_j(x,y) &= 0\,, && (13)\\
\sum_{j=1}^{n}\frac{dN_j}{dt}\Phi_j(x,y) + g\frac{\partial\zeta}{\partial x}\sum_{j=1}^{n}(h_j+\zeta_j)\,\Phi_j(x,y) &= 0\,, && (14)\\
\sum_{j=1}^{n}\frac{d\zeta_j}{dt}\Phi_j(x,y) + \frac{\partial M}{\partial x} + \frac{\partial M}{\partial y} &= 0\,. && (15)
\end{aligned}
$$

Set $(x,y)=(x_i,y_i)$. Then equations (13) - (15) become

$$
\begin{aligned}
\frac{dM_i}{dt} &= -g\,(h_i+\zeta_i)\,\frac{\partial\zeta}{\partial x}\bigg|_{(x,y)=(x_i,y_i)}\,, && (16)\\
\frac{dN_i}{dt} &= -g\,(h_i+\zeta_i)\,\frac{\partial\zeta}{\partial y}\bigg|_{(x,y)=(x_i,y_i)}\,, && (17)\\
\frac{d\zeta_i}{dt} &= -\frac{\partial M}{\partial x}\bigg|_{(x,y)=(x_i,y_i)} - \frac{\partial N}{\partial x}\bigg|_{(x,y)=(x_i,y_i)} \quad (i=1,\,2,\,\ldots,n)\,. && (18)
\end{aligned}
$$

Suppose that one of the vertices of element e_l is (x_i,y_i), and that the other two vertices are (x_j, y_j) and (x_k, y_k). Note that

$$\Phi_i(x,y) = \frac{\begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}{\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}},$$

or

$$\Phi_i(x,y) = \frac{x_j y_k - x_k y_j - x(y_k - y_j) + y(x_k - x_j)}{x_j y_k - x_k y_j - x_i(y_k - y_j) + y_i(x_k - x_j)},$$

so that the partial derivatives of $\Phi_i(x,y)$ over the element e_l are given by

$$\frac{\partial \Phi_i}{\partial x} = \frac{-(y_k - y_j)}{x_j y_k - x_k y_j - x_i(y_k - y_j) + y_i(x_k - x_j)},$$

$$\frac{\partial \Phi_i}{\partial y} = \frac{x_k - x_j}{x_j y_k - x_k y_j - x_i(y_k - y_j) + y_i(x_k - x_j)}.$$

The partial derivatives of Φ_i at the node (x_i, y_i) is determined by taking a weighted average of the partial derivatives over the elements for which the node is one of their vertices. Now standard ODE solvers are applied to the system of ordinary differential equations (16) - (18).

2 Example of tsunami simulation

Numerical techniques described in the previous section are illustrate with an example.

Example 1

Topographic data: Japan Hydrographic Association, Marine Information Research Center, MIRC-JTOPO30, M1406, 2006/09/11, Ver. 1.0.1.

Range [m]: $-300000 \le x \le 500000$, $-300000 \le y \le 300000$.

Triangular mesh: 60501 nodes, 120000 elements.

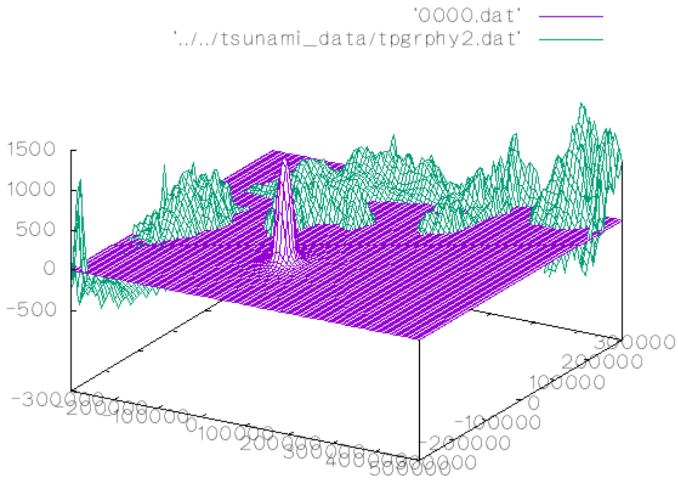

Figure 3: $t = 0$.

Initial surface displacement [m]:

$$z = \begin{cases} ae^{-\left[(x-x_c)^2+(y-y_c)^2\right]}, & (x-x_c)^2+(y-y_c)^2 \le \sigma^2\,, \\ 0\,, & (x-x_c)^2+(y-y_c)^2 > \sigma^2\,. \end{cases}$$

$x_c = 50000$, $y_c = -120000$, $a = 10$, $\sigma = 20000$.

ODE solvers: The Runge-Kutta method (the first three steps), the four step Adams-Bashforth-Moulton predictor corrector in PECE mode.

Time step [s]. 1.0.

References

[1] Mitchell, A. R., Wait, R., The Finite Element Method in Partial Differential Equations, John Wiley & Sons, London, New York, Sydney, Toronto, 1977.

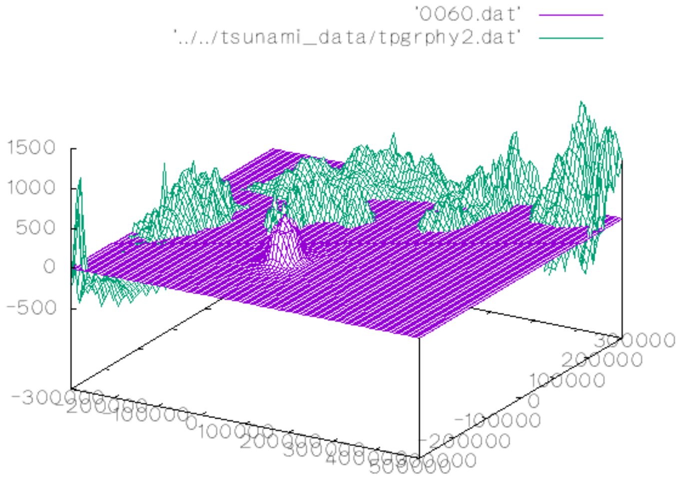

Figure 4: $t = 60$.

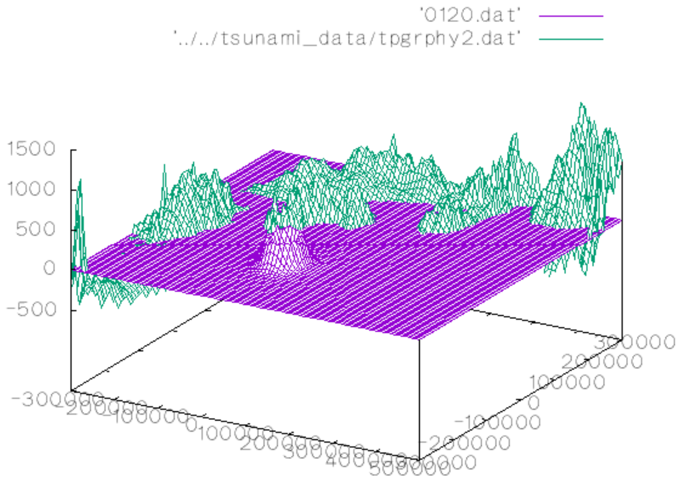

Figure 5: $t = 120$.

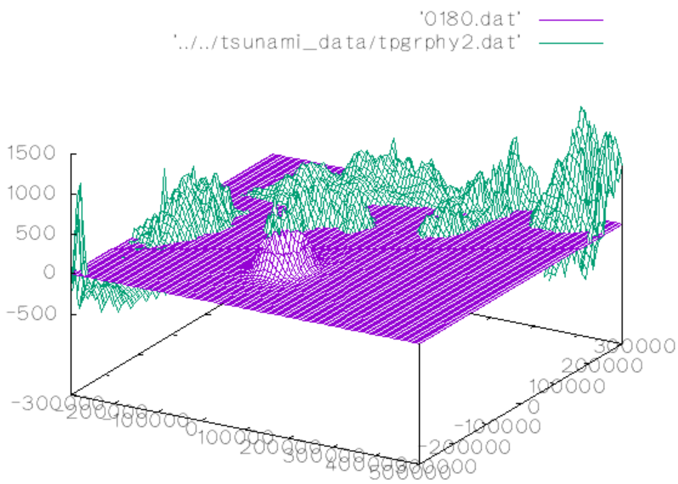

Figure 6: $t = 180$.

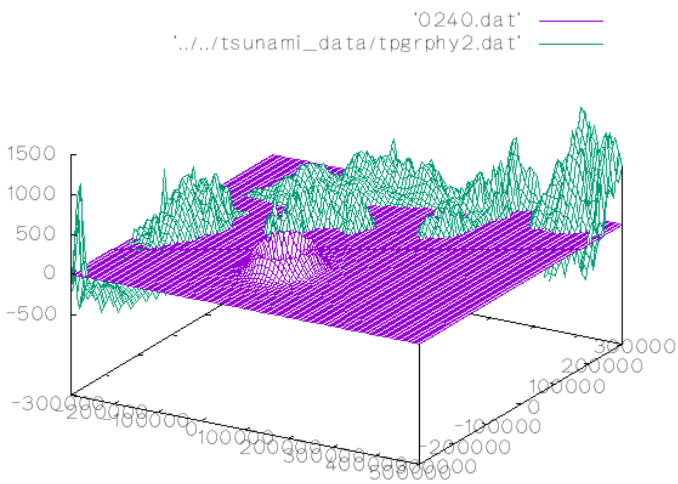

Figure 7: $t = 240$.

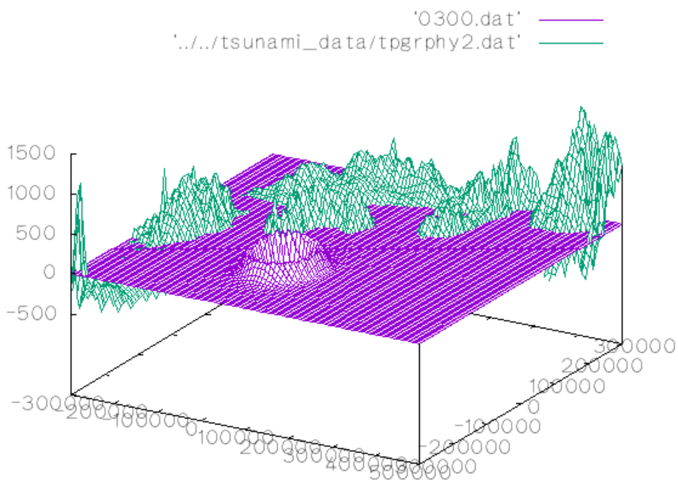

Figure 8: $t = 300$.

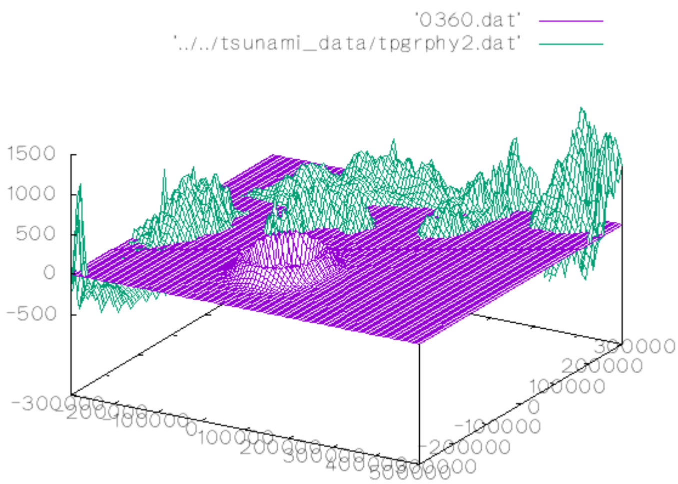

Figure 9: $t = 360$.

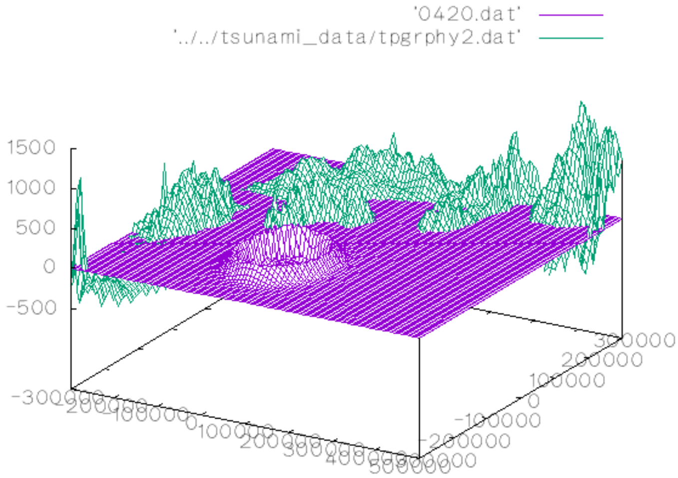

Figure 10: $t = 420$.

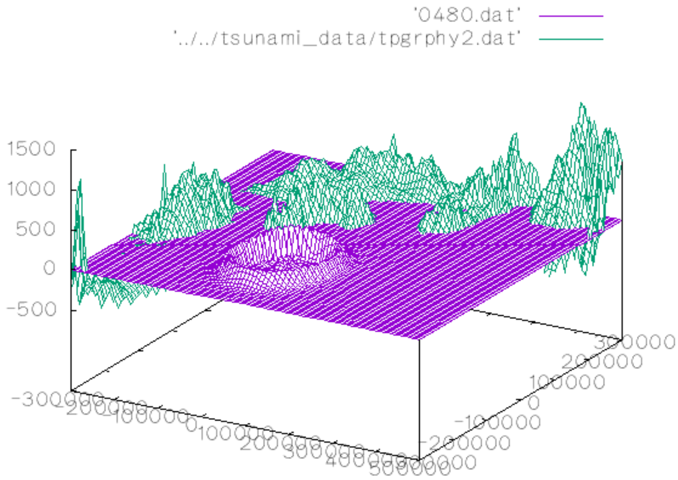

Figure 11: $t = 480$.

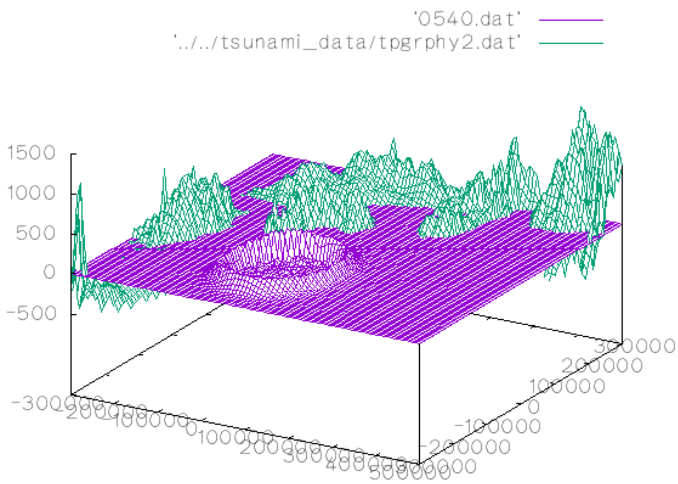

Figure 12: $t = 540$.

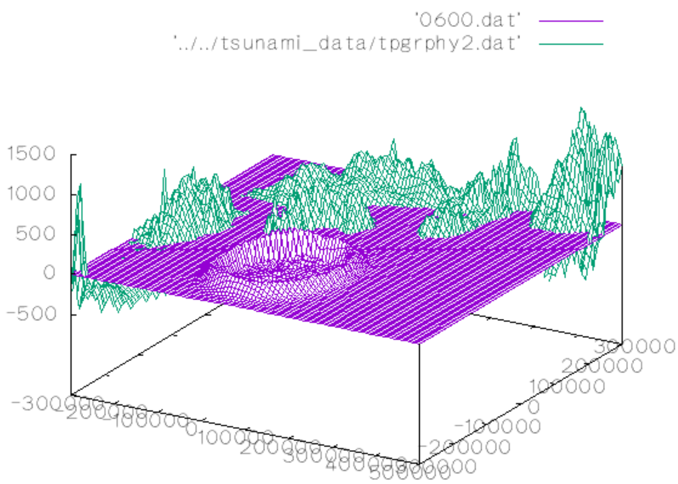

Figure 13: $t = 600$.

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