Connections between Combinatorics and Commutative Algebra

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Several applications in Chemistry, Biology, Statistics, Computer Sciences..

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Suppose that n distinct objects, each available in r identical copies, are distributed among n persons in such a way that each person receives exactly r objects. What can be said about the number $H(n, r)$ of such distributions?

The problem can be restructed as:
Let a_{ij} denote the number of copies of object i that person j receives, then $A = (a_{ij}) \in Z^{n \times n}$ is an $n \times n$ matrix such that the sum of each row and column of A is r.

$$\sum_{k=1}^{n} a_{ik} = \sum_{l=1}^{n} a_{lj} = r, \quad \text{for } i, j = 1, \ldots, n.$$

Then $H(n, r)$ is the number of such matrices A.

8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

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- there exists a polynomial $P_n(r)$ of degree $(n-1)^2$such that $H(n,r) = P_n(r)$ for all $r > n$; in particular $P_n(r) = 0$ for $r = 1, \ldots, n-1$

Let M_n be the set of solutions. Then Stanley observed the following:

- M_n has an algebraic sturucture of a submnoid of $\mathbb{Z}_+^{n\times n}$.
- The monoid algebra $K[M_n]$ has Krull dimension $(n-1)^2+1$
- By using the theory of Hilbert functions, one see that conjectures hold.

The d-dimensional cyclic polytope with n vertices is the convex hull

$$C(n, d) := conv.hull\{x(t_1), x(t_2), \ldots, x(t_n)\}$$

of $n > d \geq 2$ distinct points $x(t_i)$ on the moment curve.

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The moment curve in $\mathbb{R}_d$ is

$$x : \mathbb{R} \to \mathbb{R}^d, \quad x(t) := (t, t^2, \ldots, t^d).$$

Theorem (Upper bound "Conjecture")

The cyclic polytopes have the maximum possible number of faces for a given dimension and number of vertices. If Δ is a simplicial sphere of dimension $d-1$ with n vertices, then

$$f_i(\Delta) \leq f_i(C(n,d)), \qquad \text{for } \ i = 0, 1, \ldots, d-1$$

The conjecture was proposed in 1957.

Proved in 1970: for simplicial polytopes by Peter McMullen.

Proved in 1975: for simplicial spheres by Richard P. Stanley.

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- A monomial in S is a product of variables,

$$x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n},\ (a_1, a_2, \ldots, a_n) \in \mathbb{N}$$

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  for example, $x_2 x_4^2 x_5$.
- A squarefree monomial is a monomial in which no variable can appear twice, for example $x_2 x_4 x_5$.

Let $\mathcal{A} = \{u_1, \ldots, u_r\}$ be a set of r squarefree monomials in S. Typically, we can associate two algebraic structures of combinatorial nature related to $\mathcal{A}$.

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$\dim \Delta = \max\{|F| \ : \ F \in \Delta\} - 1$

Example: Let $\Delta =< \{1,2,3\}, \{3,4,5\}, \{2,5\}, \{1,2\}, \{2,3\}, \{1,3\}, \{3,4\}, \{3,5\}.\{4,5\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\} >$.

$\dim \Delta = 2$

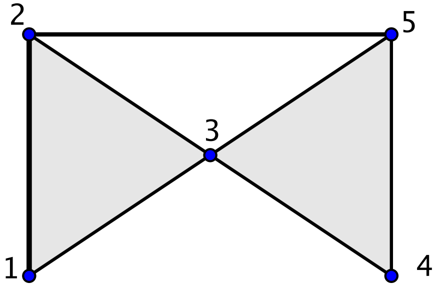

Figure: Simplicial complex

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- To each non-facet $\{i_1, i_2, \dots, i_t\}$ of Δ, attach a monomial $x_{i_1} x_{i_2} \dots x_{i_t}$.
- $I_\Delta = (x_{i_1} x_{i_2} \dots x_{i_t} \ : \{i_1, i_2, \dots, i_t\} \notin \Delta)$. Note that, because of the algebraic structure of I_Δ, it is enough to consider to the minimal non-faces.

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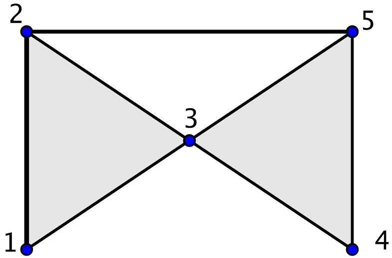
- The ideal I_Δ is called Stanley-Reisner ideal of Δ.
- $K[\Delta] = S/I_\Delta$ is called the Stanely-Reisner ring of Δ

Example: The minimal non-faces of Δ are $\{1,4\},\{1,5\},\{2,4\}$ and $\{2,3,5\}$. Then

$$I_\Delta = (x_1x_4, x_1x_5, x_2x_4, x_2x_3x_5)$$

.



- Given a simplicial complex, one attach a simplicial complex to it.
- Given a squarefree monomial ideal, one can attach a uniqe simplicial complex to it.
- Stanley-Riesner correspondence

  sqaurefree monomial ideals $\longleftrightarrow$ simplicial complexes.

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Geometrically, simplicial complexes are collections of simplices glued together by their faces.

A geometric simplicial complex Δ is a collection of simplices of $\mathbb{R}^n$ such that whenever $\sigma, \sigma' \in \Delta$, one has

- If $\tau \subset \sigma$, then $\tau \in \Delta$.
- $\sigma \cap \sigma$ is a face of both σ and σ'.

Recall that, $\dim \Delta = \max\{|F| \ : \ F \in \Delta\} - 1$

Theorem

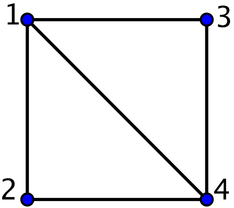
Let Δ be a simplical complex on n vertices. Then $\dim K[\Delta] = \dim \Delta + 1$.

(Graph theory $\longleftrightarrow$ Commutative algebra)

Let G be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. Then the edge ideal associated with G is

$$I(G) = (x_i x_j \ : \ \{i, j\} \in E(G)) \subset K[x_1, x_2, \ldots, x_n]$$

Let G be the following graph on vertex set $V(G) = \{1, 2, 3, 4\}$.



Then $I(G) = (x_1x_2, x_1x_3, x_1x_4, x_2x_4, x_3x_4)$.

$E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$

A graph G is chordal if every cycle of four or more vertices of G has a chord.

Theorem (Fröbergs Theorem)

Let G be a graph. Then $I(G)$ has a linear resolution if and only if G^c is a chordal graph.

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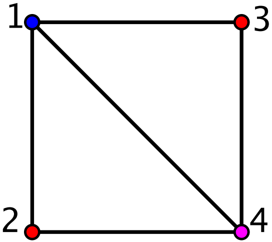
Let G be a graph. Then $I(G)$ has a linear resolution if and only if G^c is a chordal graph.

Let $I \subset S$ be a monomial ideal. Then the ideal I has a d-*linear resolution*, if I has the following minimal graded free resolution:

$$0 \to S^{\beta_t}(-(d+t)) \to \cdots \to S^{\beta_1}(-(d+1)) \to S^{\beta_0}(-d) \to I \to 0$$

.

A colouring of a graph G is an assignment of a colour to each vertex so that adjacent vertices, i.e., vertices joined by an edge receive different colours. The chromatic number of a graph G, denoted $\chi(G)$, is the minimum number of colours needed to colour G.



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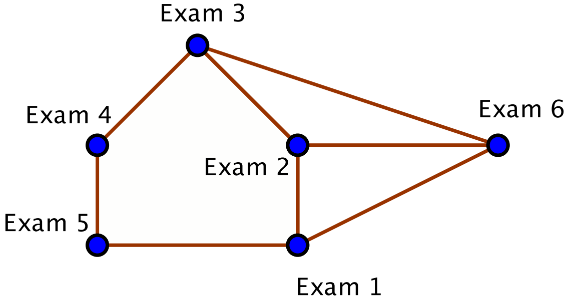
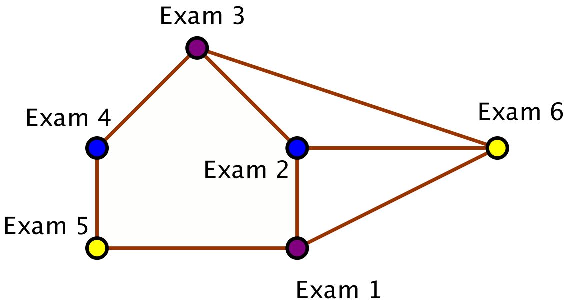
Suppose we want to schedule a set of exams. Represent each exam by a vertex, and join two vertices if there is a student who must write both exams. For example, suppose we end up with the graph:


Exam 3
Exam 4
Exam 6
Exam 2
Exam 5
Exam 1



We need 3 colors.

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$$x_W = \prod_{x_i \in W} x_i$$

$$J(G) = (x_W \ : \ W \text{ is a minimal vertex cover of } G)$$

Theorem (C.A. Francisco, H.T. Há, and A. Van Tuyl)

$$\chi(G) = \min\{d \ : \ (x_1 \ldots x_n)^{d-1} \in J(G)^d\}.$$

Thank you