Actions of Hochschild cohomology in representation theory of finite groups

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8 November 2018

Outline

- Group algebras and modules
- Categories in representation of finite groups
- Actions of the Hochschild cohomology

Group algebras

- k field, e.g $\mathbb{Q}, \mathbb{R}, \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}, \mathbb{F}_{p^n}$, etc
- G finite group, e.g. $\{1\}$, $C_p \cong \mathbb{Z}/p\mathbb{Z}$, $C_p \times C_p$, S_n, etc

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and

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g \in G} \sum_{h \in G} \lambda_g \mu_h gh$$

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(2) $G = C_2 = \{1, a\}$ with $a^2 = 1$, $kG = \{\lambda \cdot 1 + \mu \cdot a \mid \lambda, \mu \in k\}$, and

$$\begin{aligned} & (\lambda_1 \cdot 1 + \mu_1 \cdot a)(\lambda_2 \cdot 1 + \mu_2 \cdot a) \\ = & \lambda_1\lambda_2 \cdot 1 + \lambda_1\mu_2 \cdot a + \mu_1\lambda_2 \cdot a + \mu_1\mu_2 \cdot 1 \\ = & (\lambda_1\lambda_2 + \mu_1\mu_2) \cdot 1 + (\lambda_1\mu_2 + \lambda_2\mu_1) \cdot a. \end{aligned}$$

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Thus

$$kG \cong k[X]/\langle X^2 - 1\rangle, \quad \lambda \cdot 1 + \mu \cdot a \mapsto \lambda + \mu X$$

Group algebras

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In general, $G = (C_p)^n = C_p \times C_p \times \cdots C_p$ and $\mathrm{char}(k) = p$, then

$$kG \cong k[X_1, X_2, \ldots, X_n]/\langle X_1^p, X_2^p, \ldots, X_n^p \rangle$$

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Observation: kG is commutative if and only if G is Abelian, e.g. kS_3 is not commutative

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- (left) kG-module: vector space M together with 'multiplication'

  $$kG \times M \to M, \quad (a, m) \mapsto a \cdot m,$$

  satisfying the 'usual' modules axioms, e.g. $1 \cdot m = m$, $(a + b)m = am + bm$, $(ab)m = a(bm)$, $a(\lambda m + \mu n) = \lambda am + \mu an$, etc

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Example

1. The zero vector space (of dimension 0)
2. k is a kG-module with $\left(\sum_{g \in G} \lambda_g \cdot g\right) \cdot \mu = \sum_{g \in G} \lambda_g \mu$
3. kG is a kG-module with the usual multiplication

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- Composition of two kG-homomorphisms is again a kG-homomorphism, and

  the class of all kG-modules together with kG-homomorphism between them form a category, denoted by $\mathrm{Mod}(kG)$.

Definition of categories

A category $\mathcal{C}$ consists of

- a 'class' of objects $\mathrm{Ob}(\mathcal{C})$
- a set of morphism $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$
- a function, called composition

$$\begin{aligned} \circ\colon\ \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) &\to \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

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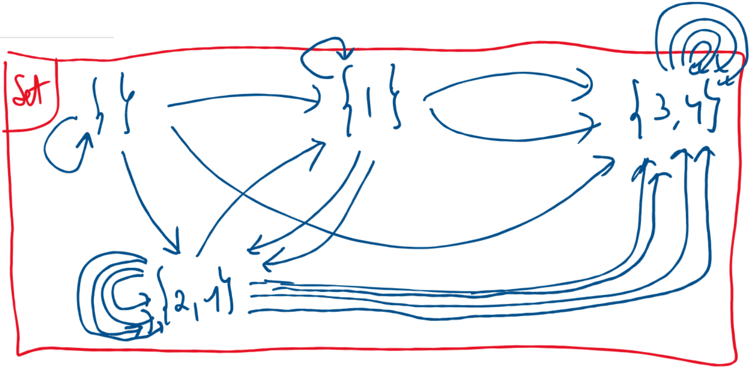
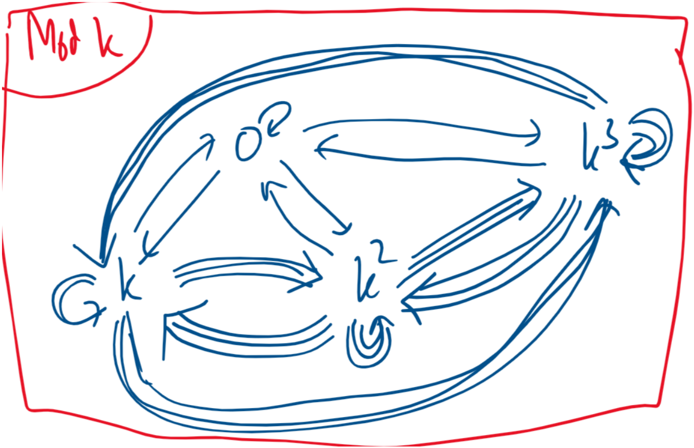
A 'morphism' between categories is called a functor

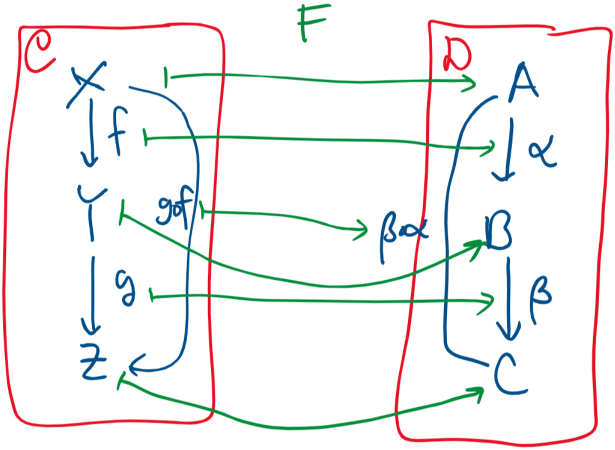
Category Set



Category Mod(k)



Functor



$$F(g\circ f) = F(g)\circ F(f)$$

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- In 'modular representation theory':
  If $\mathrm{char}(k) \mid |G|$, the group algebra kG is usually of 'wild representation type', hence it is impossible to classify the indecomposables modules.

Categories in modular representation theory

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1. Stable module category $\mathrm{StMod}(kG)$ or $\underline{\mathrm{Mod}}(kG)$
   - objects: kG-modules
   - morphisms: $\underline{\mathrm{Hom}}_{kG}(M, N) = \mathrm{Hom}(M, N)/\,\mathrm{PHom}(M, N)$,

$$\mathrm{PHom}(M, N) = \{f \mid f \text{ factors through some proj. ob.}\}$$

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    - objects: 'complexes' of kG-modules
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3. Homotopy category of injective kG-modules $\mathrm{K}(\mathrm{Inj}\, kG)$

Homotopy category of injectives

- Objects: complexes of injective kG-modules

$$\cdots \longrightarrow I^{n-1} \xrightarrow{d^{n-1}} I^{n} \xrightarrow{d^{n}} I^{n+1} \longrightarrow \cdots$$

where I^n are injective kG-modules and $d^n \circ d^{n-1} = 0$.

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- Morphism: chain maps (of degree 0) modulo null-homotopy maps

$$\mathrm{Hom}_{\mathsf{K}(\mathrm{Inj}\, kG)}(I, J) = \frac{\{\text{chain maps } I \to J\}}{\{\text{null-homotopic maps } I \to J\}}$$

Morphisms in homotopy category of injectives

$$
\left\{
\begin{tikzcd}
\cdots \arrow[r] & I^{n-1} \arrow[r, "d_I^{n-1}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & I^n \arrow[r, "d_I^n"] \arrow[d, "f^n"] \arrow[dr, phantom, "\circlearrowright"] & I^{n+1} \arrow[r] \arrow[d] & \cdots \\
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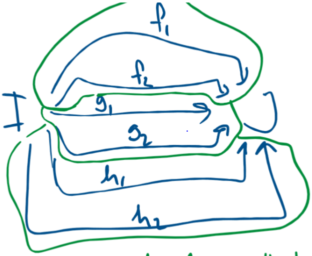
Morphisms in homotopy category of injectives



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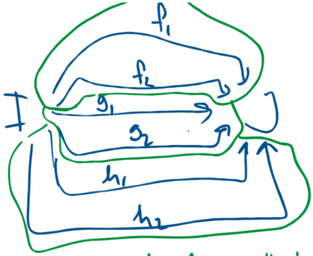

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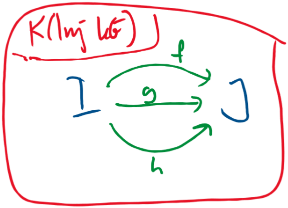

Homotopy category of injectives

Theorem (Krause 2005)

There exists a diagram of six functors

$$\begin{tikzcd} \underline{\mathrm{Mod}}\, kG \arrow[r, shift left=3, leftarrow] \arrow[r] \arrow[r, shift right=3, leftarrow] & K(\mathrm{Inj}\, kG) \arrow[r, shift left=3, leftarrow] \arrow[r] \arrow[r, shift right=3, leftarrow] & D(\mathrm{Mod}\, kG) \end{tikzcd}$$

satisfying some 'nice' properties.

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Remark

The category $K(\mathrm{Inj}\,kG)$ contains a copy of $\underline{\mathrm{Mod}}(kG)$ and two copies of $\mathrm{D}(\mathrm{Mod}(kG))$

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- Plan: Give some algebraic structure on the set of morphism between two objects

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Benson-Iyengar-Krause theory (2008):
If R is a graded-commutative notherian ring 'acting' on $\mathsf{K}(\mathsf{Inj}\, kG)$ then there is a 'local cohomology' functor

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Define the support of an object X in $\mathsf{K}(\operatorname{Inj} kG)$ to be

$$\operatorname{supp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}^h(R) \mid \Gamma_{\mathfrak{p}}(X) = 0\}.$$

Some homological algebras

A an f.d. algebra over a field k and X, Y A-modules

- an injective resolution of X is a complex of injective modules

$$iX = \cdots \to 0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

  together with a map $\eta\colon X \to I^0$ such that the sequence

$$0 \to X \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

  is exact.
- In particular, iX is an object in $\mathrm{K}(\mathrm{Inj}\, A)$

Some homological algebras

- The n-th extension group $\operatorname{Ext}^n(X, Y)$ is

$$\operatorname{Ext}^n_A(X, Y) = \operatorname{Hom}_{\mathsf{K}(\operatorname{Inj} A)}(iX, iY[n]),$$

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$$\begin{tikzcd}
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\cdots \arrow[r] & J^{i-1+n} \arrow[r, "(-1)^n d_J"'] & J^{i+n} \arrow[r, "(-1)^n d_J"'] & J^{i+1+n} \arrow[r] & \cdots
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- The Hochschild cohomology ring of A is the graded ring

$$\mathrm{HH}^*(A, A) = \bigoplus_{n \in \mathbb{Z}} \mathrm{HH}^n(A, A)$$

with multiplication defined using composition and shift.

Hochschild cohomology

- If $f \in \mathrm{HH}^m(A, A)$, i.e. $f\colon iA \to iA[m]$
  and $g \in \mathrm{HH}^n(A, A)$, i.e. $g\colon iA \to iA[n]$,

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  then $f \cdot g$ is defined as the composition

  $$iA \xrightarrow{f} iA[m] \xrightarrow{g[m]} iA[m+n]$$

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Theorem [Gerstenhaber 1963]. The Hochschild cohomology ring of A is a graded-commutative ring.

Theorem [Evens 1961, Venkov 1959, Ginzburg-Kumar 1993, Friedlander-Suslin 1997]. The Hochschild cohomology ring of A is noetherian.

The Action

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- Idea: Use tensor product over A
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- Restrict to the homotopy category of injectives

$$- \otimes_A - \colon \ \mathsf{K}(\operatorname{Inj} A^e) \times \mathsf{K}(\operatorname{Inj} A) \to \mathsf{K}(\operatorname{Inj} A)$$

The Action

k field and G finite group, $A = kG$

- Idea: Use tensor product over A
  If X is an (A, A)-bimodule and M an A-module, then $X \otimes_A M$ is an A-module.
- Extend to tensor product on the homotopy category:

$$- \otimes_A - \colon \; \mathsf{K}(\operatorname{Mod} A^e) \times \mathsf{K}(\operatorname{Mod} A) \to \mathsf{K}(\operatorname{Mod} A)$$

- Restrict to the homotopy category of injectives

$$- \otimes_A - \colon \; \mathsf{K}(\operatorname{Inj} A^e) \times \mathsf{K}(\operatorname{Inj} A) \to \mathsf{K}(\operatorname{Inj} A)$$

The Action

$A = kG$, $\mathsf{K} = \mathsf{K}(\operatorname{Inj} A)$,

$$\operatorname{Hom}^*_{\mathsf{K}}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}^*_{\mathsf{K}}(X, Y[n])$$

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- Apply the bifunctor to the morphisms, get:

$$\mathrm{HH}^*(A, A) \times \operatorname{Hom}^*_{\mathsf{K}}(X, Y) \to \operatorname{Hom}^*_{\mathsf{K}}(X, Y),$$

where a 'homogeneous' pair $f\colon iA \to iA[m]$ and $g\colon X \to Y[n]$ is mapped to

$$X \cong iA \otimes_A X \xrightarrow{f \otimes g} iA[m] \otimes_A Y[n] \cong Y[m+n]$$

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- Get (graded) $\mathrm{HH}^*(A, A)$-module structure on $\operatorname{Hom}^*_{\mathsf{K}}(X, Y)$ for each pair $X, Y \in \mathsf{K} = \mathsf{K}(\operatorname{Inj} A)$.

Main Theorem

Theorem (Theorem 4.3.7). *Let A be a finite dimensional self-injective algebra over $\mathcal{K}$. Then the tensor triangulated category $\mathcal{K}(\operatorname{Inj} A^e)$ acts on $\mathcal{K}(\operatorname{Inj} A)$ via tensor product of complexes over A. In particular, we get an action of $\mathrm{HH}^*(A/\Bbbk)$ on $\mathcal{K}(\operatorname{Inj} A)$ given by*

$$\mathrm{HH}^*(A/\Bbbk) \cong \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A^e)}(\mathbf{i}A) \xrightarrow{-\otimes_A M} \operatorname{End}^*_{\mathcal{K}(\operatorname{Inj} A)}(M)$$

for all $M \in \mathcal{K}(\operatorname{Inj} A)$

Conclusion and Outlook

- The Hochschild cohomology ring acts on the homotopy category of injective kG-modules
- Since the stable category and the derived category is 'contained' in the homotopy category of injectives, we get also actions on both categories (for free)
- The above action can be generalized to arbitrary f.d. self-injective algebra over a field k
- Using B-I-K's machinery, get local cohomology functor and support theory for all categories above

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An open problem:

- Does the above action generalize to arbitrary f.d. algebras?

Thank You!